CHAPTER 3 - FIVE-DIMENSIONAL SYSTEMS

During the preceding development of the Dynamic Theory, there did not appear to be anything that approached a description of nuclear effects. Of course quantum theorists may respond that the nuclear effects lie within the realm of quantum theory. This, however, does not seem to be a strong argument since current nuclear theory appears to depend upon a number of ad hoc postulates.

If it is supposed that nuclear theory cannot be extracted from some aspect of the preceding four-dimensional world view, then how might the Dynamic Theory produce a foundation for nuclear theory? At this point there may appear to be no obvious way. Therefore, let us proceed on a different tack.

Thus far we have constantly adhered to the policy of dividing systems into two types: thermodynamic systems with only a work term of the pdv type and mechanical systems with three mechanical, or spatial, work terms. The generality of the adopted laws places no restrictions upon the number or type of variables used. In particular, there is no restriction coming from the laws themselves which says we cannot use four work terms, one the thermodynamic pdv term and three mechanical Fdq terms. Obviously pdv itself is just another Fdq type term with the pressure as the generalized force and the volume as the generalized displacement.

When we teach thermodynamics we write the First Law with the right hand side of the equation being the change in internal energy (system energy), the thermodynamic work term, and three spatial work terms. We then tell the students that since the right hand side of the equation involves five unknowns we must have five independent equations in order to have a solvable system. The first equation offered is the conservation of mass which we state guarantees that we may write mass density as a function of space and time. But is this really true for all space and for all time?

The rub comes in attempting to visualize a world description in five dimensions. Many arguments may be envisioned which tend to imply only a four-dimensional manifold is needed. The kinetic theory of gases relates the pressure to the average velocities of the particles contained. Does that not imply that thermodynamics ultimately rests on a four-dimensional manifold? Recall that the system in the kinetic theory is basically in equilibrium.

Statistical thermodynamists may claim that thermodynamics is basically statistical in nature and is fundamentally tied to order and disorder and hence to the four-dimensional world of quantum theory. But remember that the overall system, to which the statistical approach is applicable, is a composite system made up of many subsystems each in an equilibrium state. What happens to this argument when the number of individual particles is not infinitely large?

Still there seems to be no substantial support for a five-dimensional world from the point of view of current theories. This is to be expected though in view of the difficulties experienced in the transition from the classical three-dimensional world to the four-dimensional space-time of Einstein's theories. Obviously had the extension of the universe been restricted on a priori grounds to three-dimensional Euclidean space, Einstein's theory would have been rejected on first principles. On the other hand, as soon as we recognize that the fundamental continuum of the universe and its geometry cannot be posited a priori and can only be disclosed to us from place to place by experiment and measurement, a vast number of possibilities are thrown open. Among these the four-dimensional space-time of relativity, with its varying degrees of non-Euclideanism, has found a place. So also may the five-dimensional view of the Dynamic Theory be found within the possibilities. Ultimate judgment upon its necessity, or applicability, should rest upon a comparison of the theory's predictions with reality.

A. Systems Near an Equilibrium State

The metric coefficients are made up of the second partial derivatives of the system energy function and, therefore, if the system remains near an equilibrium state, then the value of these derivatives evaluated at the equilibrium state may be used as a first approximation for the metric coefficients. In this case the geometry will be Euclidean and, from the preceding four-dimensional development, the Euclidean manifold produced by applying the E-conservative restriction was Minkowski's space-time continuum of Special Relativity.

Therefore, suppose we begin an investigation of the five-dimensional world by staying very near an equilibrium state so as to simplify the description to a five-dimensional generalization of Minkowski's space-time manifold.

3.1 Equations of Motion

Suppose that we consider some sort of system requiring four work terms and for the moment not concern ourselves as to exactly what this system might be. Thus, for our system we will have thermodynamic as well as mechanical variables and the First Law becomes

$$\_\tilde{E} = d\tilde{U} + Pdv - \tilde{F}_\alpha dq^\alpha ; \quad \alpha = 1,2,3.$$

Where the v and are considered as specific quantities. That is, these quantities are related to a unit of mass such as is customary in thermodynamics.

The specific volume is the reciprocal of the mass density γ. Using the mass density instead of the specific volume the First Law becomes

$$\overline{d}\widetilde{E} = d\widetilde{U} - \left(\frac{P}{\gamma^2}\right)d\gamma - \widetilde{F}_\alpha dq^\alpha ; \quad \alpha = 1,2,3.$$

This law now requires that the system's specific energy U be a function of five independent variables so that U = U(S,q^1,q^2,q^3). Thus, the First Law requires a five-dimensional manifold of specific entropy, space, and mass density for a general system. Since the system under consideration needs both thermodynamic and mechanical variables, we can no longer refer to the entropy as mechanical or thermodynamic; however, the limiting case where the mass is held fixed must produce the mechanical entropy.

The procedure established by the Dynamic Theory is to take the stability condition quadratic form as the metric for a stable system. Thus, the coefficients of the metric become the second partial derivatives of the energy function. In order to simplify the metric, suppose for the present that we restrict our system to be very near an equilibrium state so that we may consider the second partial derivatives to be constants. This is in essence considering a local Euclidean manifold; the symmetry of the geometric connections guarantees that we may do this.

Since the metric coefficients are constants, a transformation may be found such that the cross terms are zero. Then in this coordinate system and when

$$q^0 \equiv \frac{S}{F_0} \quad and \quad q^4 \equiv \frac{\gamma}{a_0},$$

the metric becomes

$$c^2 (dt)^2 = (dq^0)^2 + dq^\alpha dq^\alpha + (dq^4)^2 ; \quad \alpha = 1,2,3. \tag{3.1}$$

If we again consider the restriction d = 0 so that we are talking of an E-conservative system for which the principle of increasing entropy holds, then we have the variational principle given by

$$\delta \int \sqrt{(dS)^2} = 0. \tag{3.2}$$

Solving Eqn. (3.1) for dq^0 and squaring we get

$$(dq^0)^2 = c^2 (dt)^2 - dq^\alpha dq^\alpha - (dq^4)^2 ; \quad \alpha = 1,2,3, \tag{3.3}$$

or

$$\left(\frac{dq^0}{dt}\right)^2 = c^2 - g_{\alpha\beta}\left(\frac{dq^\alpha}{dt}\right)\left(\frac{dq^\beta}{dt}\right); \quad \alpha, \beta = 1,2,3,4.$$

The entropy manifold given by Eqn. (3.3) is a five-dimensional Minkowski-type manifold with coordinates of space-time-mass. We may, therefore, follow the procedure Minkowski and Einstein used in the Special Theory of Relativity.

First, to avoid confusion, let us rename the coordinates as

$$x^0 \equiv ct,\ \ x^1 \equiv q^1,\ \ x^2 \equiv q^2,\ \ x^3 \equiv q^3,\ \ and\ \ x^4 \equiv q^4.$$

Then define the five-dimensional velocity vector as

$$u^i \equiv \frac{dx^i}{dq^0};\quad i = 0,1,2,3,4$$

and define the five-dimensional acceleration vector as

$$f^i \equiv \frac{\delta u^i}{\delta q^0} \equiv \frac{d^2 x^i}{dq^{0^2}} + \begin{pmatrix} i \\ jk \end{pmatrix} \frac{dx^j}{dq^0}\frac{dx^k}{dq^0}.$$

Now the specific entropy is the arc length and the variational principle is based upon the entropy. Therefore, if we multiply the specific entropy by the mass density, we have the entropy density. The variational problem becomes

$$\delta \int \sqrt{\gamma^2 (dq^0)^2} = \delta \int \gamma \sqrt{(dq^0)^2} = 0. \tag{3.4}$$

Notice how the mass has entered our variational problem. It has entered because our metric was in terms of the "specific entropy", or entropy per unit mass. The variational problem is based upon the entropy, not the specific entropy. Thus, the mass density is required in the variational problem to correct this difference. The importance of this lies in the fact that this is the origin of the "inertia" which appears in the following equations of motion.

The Euler equations for this problem are

$$\frac{d}{dq^0}\left[\frac{\gamma\, g_{ij} u^j}{\sqrt{g_{ij}u^i u^j}}\right] - \frac{\partial \gamma}{\partial x^k}\sqrt{g_{ij}u^i u^j} - \left[\frac{\gamma \dfrac{\partial g_{ik}}{\partial x^k} u^i u^j}{\sqrt{g_{ij}u^i u^j}}\right] = 0$$

or

$$a^0 u^4 \left[\frac{g_{ij} u^j}{\sqrt{g_{ij}u^i u^j}}\right] - \frac{\partial \gamma}{\partial x^i}\sqrt{g_{ij}u^i u^j} + \left\{\frac{d}{dq^0}\left[\frac{g_{ij} u^j}{\sqrt{g_{ij}u^i u^j}}\right] - \frac{\dfrac{\partial g_{ik}}{\partial x^i} u^i u^j}{\sqrt{g_{ij}u^i u^j}}\right\} = 0.$$

Using the fact that $g_{ij}u^i u^j = 1$, the Euler equations become

$$\gamma f^i = \frac{\partial\gamma}{\partial x^i} - a_0 u^4 g_{ij} u^j \equiv F^i \tag{3.5}$$

where the F^i are force densities.

Obviously if we hold the mass density fixed, $u^4=0$, then the volume integral of this equation becomes the force-mass- acceleration relationship of Special Relativity.

Now Since

$$f^i = \frac{\delta u^i}{\delta q^0} \quad and \quad \left(\frac{dq^0}{dt}\right)^2 = c^2 - u^\alpha u^\alpha ; \quad \alpha = 1,2,3,4$$

then

$$F^i = \gamma \frac{\delta u^i}{\delta q^0} = \gamma \frac{\delta u^i}{\delta t} \frac{dt}{dq^0}$$

$$= \frac{\gamma}{\sqrt{c^2 - v^2}} \frac{\delta}{\delta t}\left(\frac{dx^i}{dq^0}\right) \quad where \ v^2 = u^\alpha u^\alpha : \alpha = 1,2,3,4.$$

Then

$$F^\alpha = \frac{\gamma}{\sqrt{c^2 - v^2}} \frac{\delta}{\delta t}\left(\frac{1}{\sqrt{c^2 - v^2}} \frac{dx^\alpha}{dt}\right)$$

$$= c^2 \frac{\gamma}{\sqrt{1 - \beta^2}} \frac{\delta}{\delta t}\left(\frac{1}{\sqrt{1 - \beta^2}} \frac{dx^\alpha}{dt}\right),$$

where ß = v/c with v being the four-dimensional speed.

The force density equation may now be written as

$$\sqrt{1 - \beta^2}\, F^\alpha = \frac{\gamma}{c^2} \frac{\delta}{\delta t}\left(\frac{1}{\sqrt{1 - \beta^2}} \frac{dx^\alpha}{dt}\right).$$

Consider

$$\frac{\delta}{\delta t}\left[\frac{\gamma}{\sqrt{1 - \beta^2}} \frac{dx^\alpha}{dt}\right] = \frac{\delta\gamma}{dt}\left[\frac{1}{\sqrt{1 - \beta^2}} \frac{dx^\alpha}{dt}\right] + \gamma \frac{\delta}{\delta t}\left[\frac{1}{\sqrt{1 - \beta^2}} \frac{dx^\alpha}{dt}\right];$$

but $\delta\gamma/\delta t = a_0 v^4$, so that the force density equations may be written as

$$\sqrt{1-\beta^2}\,F^\alpha = \frac{1}{c^2}\frac{\delta}{\delta t}\left[\frac{\gamma}{\sqrt{1-\beta^2}}\frac{dx^\alpha}{dt}\right] - \frac{a_0 u^4}{c^2}\left[\frac{1}{\sqrt{1-\beta^2}}\frac{dx^\alpha}{dt}\right].$$

We may define

$$\gamma^1 \equiv \frac{\gamma}{\sqrt{1-\beta^2}}$$

as the effective mass density or "relativistic" mass density; then

$$\sqrt{1-\beta^2}\,F^\alpha = \frac{1}{c^2}\frac{\delta}{\delta t}\left[\gamma^1\frac{dx^\alpha}{dt}\right] - \frac{a_0 v^4 v^\alpha}{c^2\sqrt{1-\beta^2}}.$$

By defining $F^\alpha \equiv c^2\sqrt{1-\beta^2}\,\hat{F}^\alpha$ 23 so that

$$\hat{F}^\alpha = \frac{\delta}{\delta t}\left[\gamma^1\frac{dx^\alpha}{dt}\right] - \frac{a_0 v^4 v^\alpha}{\sqrt{1-\beta^2}} \tag{3.6}$$

we see that this force density becomes Einstein's special relativistic force density when $v^4 = 0$, or for constant "rest mass." Thus, the equations of motion, Eqn.s (3.6), reduce to Einstein's special relativistic equations of motion when $d\gamma/dt=0$.

3.2 Energy Equation

Now for our system the restriction that

$$\_\tilde{E} = 0 = d\tilde{U} - \frac{P}{\gamma^2}d\gamma - \tilde{F}_\alpha dx^\alpha, \quad \alpha = 1,2,3$$

requires that

$$d\tilde{U} = \frac{p}{\gamma^2}d\gamma + \tilde{F}_\alpha dx^\alpha, \quad \alpha = 1,2,3,$$

or if p/γ^2 is considered as another generalized force density, then

$$d\tilde{U} = \tilde{F}_\alpha dx^\alpha, \quad \alpha = 1,2,3,4.$$

Thus, by integrating the expression for the system's specific energy change, we should arrive at the Einstein energy equation if we hold dx^4/dt / 0. Therefore, we shall perform the integration using the force densities given by Eqn. (3.6) to get the system's energy, or

$$\tilde{U} - \tilde{U}_0 = \int_{p_0}^{p} \tilde{F}_\alpha \, dx^\alpha = \int_{p_0}^{p} \left\{ \frac{d}{dt}\left[\frac{\gamma}{\sqrt{1-\beta^2}} \frac{dx^\alpha}{dt} \right] - \frac{a_0 u^4 u^\alpha}{\sqrt{1-\beta^2}} \right\} dx^\alpha$$

$$= \int_{t_0}^{t} \left\{ \gamma \left[\frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}} \right) u^\alpha u^\alpha + \frac{u^\alpha \frac{du^\alpha}{dt}}{\sqrt{1-\beta^2}} \right] \right\} dt.$$

But $c^2ß^2 = u^\alpha u^\alpha$ and $c^2ß(d/dt) = u^\alpha \, du^\alpha/dt$; therefore,

$$\tilde{U} - \tilde{U}_0 = \int_{t_0}^{t} \left\{ \gamma \left[\frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}} \right) c^2 \beta^2 + \frac{c^2 \beta \dot{\beta}}{\sqrt{1-\beta^2}} \right] \right\} dt$$

$$= c^2 \int_{\beta_0}^{\beta} \gamma \frac{\beta d\beta}{(1-\beta^2)^{\frac{3}{2}}}.$$

Now ß depends upon u and not upon x^4 or γ; therefore

$$\tilde{U} - \tilde{U}_0 = \frac{\gamma c^2}{\sqrt{1-\beta^2}}$$

or

$$\tilde{U} = \frac{\gamma c^2}{\sqrt{1-\beta^2}} + constant.$$

If the internal energy is considered to be the system's energy when the spatial velocities u^α; α = 1, 2, 3 are taken as zero, then the internal energy density given by

$$\tilde{U} = \frac{\gamma c^2}{\sqrt{1-\left(\frac{u^4}{c} \right)}} + constant.$$

At the condition where u^4 is also zero the internal energy density is then

$$\tilde{U} = \gamma c^2 + constant.$$

By taking the constant of integration to be zero, this internal energy density then is seen to correspond to Einstein's "rest energy" where here the "rest energy" is in terms of a four-dimensional "at rest" state.

If we make the usual approximation of allowing $ß^2 << 1$, then the system's energy density is approximately given by

$$\tilde{U} = \gamma c^2 + \frac{1}{2}\gamma v^2 + \frac{1}{2}\frac{\gamma}{(a_0)^2}(\dot{\gamma})^2 ,$$

where here $u^4 = d\gamma/dt$ is used. This displays the classical limit system energy density for an E-conservative system very near equilibrium.

<u>B. Systems With Non-Euclidean Manifold</u>

Suppose now we relax the assumption that the system is very near an equilibrium point so that the second partial derivatives are no longer constants but are functions. This is essentially the same transition as Einstein made going from his Special to General theory; however, the logic of the transition is much simpler here. The only change in the logic appears in the relaxation of the assumption of nearness. There is, of course, a drastic increase in mathematical difficulty since the metric components are no longer constants.

3.3 General Variational Principle

We shall consider a system that must be described by both thermodynamic and mechanical variables. When written in terms of the mass density, the First Law for this system may be written as

$$\_\tilde{E} = d\tilde{U} - \frac{P}{\gamma^2} d\gamma - \tilde{F}_\alpha dq^\alpha , \quad \alpha = 1,2,3,$$

where the tilde denotes specific quantities.

Following the prescribed procedures of the Dynamic Theory we shall take the stability condition quadratic form as the metric for our system. Thus, the metric coefficients will be given by the second partial derivatives

$$h_{ij} = \frac{\partial^2 U}{\partial q^i \partial q^j}, \quad i, j = 0,1,2,3,4,$$

where $q^4 = \gamma/a_0$. The metric may then be written as

$$c^2 (dt)^2 = h_{00}(dq^0)^2 + 2 h_{0\alpha} dq^0 dq^\alpha + h_{\alpha\beta} dq^\alpha dq^\beta , \tag{3.3A}$$

where α,ß=1,2,3,4.

Imposing the restriction that the system be E-conservative, $\pi E=0$, results in the principle of increasing entropy, so that

$$\delta \int \sqrt{(dS)^2} = 0.$$

In terms of the specific entropy the variational principle may be written as

$$\delta \int \sqrt{(\gamma\, dq^0)^2} = \delta \int \gamma \sqrt{(dq^0)^2} = 0. \tag{3.4}$$

Solving the metric given by Eqn. (3.3A) and squaring yields the expression

$$(dq^0)^2 = \left[\frac{1}{h_{00}}\right]\{c^2 (dt)^2 + 2\, h_{0\alpha}\, A dt dq^\alpha - h_{\alpha\beta}\, dq^\alpha dq^\beta\}, \quad \alpha, \beta = 1,2,3,4, \tag{3.7}$$

with

$$A = \frac{h_{0\eta}}{h_{00}} \dot{q}^\eta \pm \sqrt{\frac{c^2}{h_{00}} - \frac{h_{\eta\nu}}{h_{00}} \dot{q}^\eta \dot{q}^\nu + \frac{(h_{0\eta} \dot{q}^\eta)^2}{(h_{00})^2}}.$$

This metric in a five-dimensional manifold of space-time-mass may be rewritten as

$$(dq^0)^2 = \left[\frac{1}{h_{00}}\right](d\sigma)^2,$$

where

$$(dq^0)^2 \equiv \hat{g}_{ij}\, dx^i dx^j, \quad i, j = 0,1,2,3,4,$$

and

$$(d\sigma)^2 \equiv q'_{ij}\, dx^i dx^j, \quad i, j = 0,1,2,3,4,$$

with $x^0=ct$, $x^1=q^1$, $x^2=q^2$, $x^3=q^3$, and $x^4=\gamma/a_0$. Thus we may write

$$(dq^0)^2 = \hat{q}_{ij}\, dx^i dx^j = \left(\frac{1}{f}\right)(d\sigma)^2 = \left(\frac{1}{f}\right) g'_{ij}\, dx^i dx^j. \tag{3.8}$$

Having established the metrics in Eqn. (3.8) in the manner prescribed by the Dynamic Theory, the geometry must be Weyl geometry; wherein the potential five-vector is defined as

$$\phi_i \equiv \pm \frac{\partial \ln f^{\frac{1}{2}}}{\partial x^i} \tag{3.9}$$

and the field tensor is given by

$$F_{ij} \equiv \phi_{i,j} - \phi_{j,i}. \tag{3.10}$$

3.4 Gauge Function Field Equations

In order to isolate the field equations resulting from a gauge function from the field equations produced by a vector curvature, let us consider a local Euclidean manifold for $(d\sigma)^2$.

Now the field tensor given by Eqn. (3.10) has 25 components. We would like to determine the field equations for these components. The quickest, though not the only, way is to consider the five dimensions to be $x^0=ict$, $x^\alpha=x^\alpha$, $\alpha=1,2,3,4$. The field tensor is then defined to be

$$F_{ij} = \begin{vmatrix} 0 & iE_1 & iE_2 & iE_3 & iV_4 \\ -iE_1 & 0 & B_3 & -B_2 & V_1 \\ -iE_2 & -B_3 & 0 & B_1 & V_2 \\ -iE_3 & B_2 & -B_1 & 0 & V_3 \\ -iV_4 & -V_1 & -V_2 & -V_3 & 0 \end{vmatrix}.$$

Using Bianchi's identities

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0$$

and the various combinations of the indices 0, 1, 2, 3, 4 we obtain the field equations

$$\begin{aligned} \bar{\Delta} \bullet \bar{B} &= 0 \\ \bar{\Delta} x \bar{E} + \frac{1}{c}\frac{\partial \bar{B}}{\partial t} &= 0 \\ \bar{\Delta} x \bar{V} + a_0 \frac{\partial \bar{B}}{\partial \gamma} &= 0 \\ \bar{\Delta} V_4 + \frac{1}{c}\frac{\partial \bar{V}}{\partial t} + a_0 \frac{\partial \bar{E}}{\partial \gamma} &= 0. \end{aligned} \tag{3.11}$$

The definition of the five-vector current density

$$\frac{\partial F_{ij}}{\partial x^i} \equiv \frac{4\pi}{c} J_i \tag{3.12}$$

yields the equations

$$\begin{aligned} \bar{\Delta} \bullet \bar{E} + a_0 \frac{\partial V_4}{\partial \gamma} &= 4\pi\rho \\ \bar{\Delta} x \bar{B} - \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + a_0 \frac{\partial \bar{V}}{\partial \gamma} &= \frac{4\pi \bar{J}}{c} \\ \bar{\Delta} \bullet \bar{V} + \frac{1}{c}\frac{\partial V_4}{\partial t} &= -\frac{4\pi J_4}{c}. \end{aligned} \tag{3.13}$$

In addition to these field equations there is the statement of conservation of charge where

$$\frac{\partial J_i}{\partial x^i} = 0, \quad i = 0,1,2,3,4,$$

so that

$$\frac{\partial \rho}{\partial t} + \overline{\Delta} \bullet \overline{J} + a_0 \frac{\partial J_4}{\partial \gamma} = 0. \tag{3.14}$$

For ease in future reference to these eight field equations they may be rewritten as

$$\begin{aligned}
&\overline{\Delta} \bullet \overline{B} = 0 && [a] \\
&\frac{1}{c}\frac{\partial \overline{B}}{\partial t} + \overline{\Delta} x \overline{E} = \overline{0} && [b] \\
&\overline{\Delta} x \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} + a_0 \frac{\partial \overline{V}}{\partial \gamma} = \frac{4\pi \overline{J}}{c} && [c] \\
&\overline{\Delta} \bullet \overline{E} + a_0 \frac{\partial V_4}{\partial \gamma} = 4\pi\rho && [d] \\
&\frac{\partial \rho}{\partial t} + \overline{\Delta} \bullet \overline{J} + a_0 \frac{\partial J_4}{\partial \gamma} = 0 && [e] \\
&\overline{\Delta} x \overline{V} + a_0 \frac{\partial \overline{B}}{\partial \gamma} = \overline{0} && [f] \\
&\overline{\Delta} V_0 + \frac{1}{c}\frac{\partial \overline{V}}{\partial t} = a_0 \frac{\partial \overline{E}}{\partial \gamma} && [g] \\
&\overline{\Delta} \bullet \overline{V} + \frac{1}{c}\frac{\partial V_4}{\partial t} = -\frac{4\pi J_4}{c} && [h]
\end{aligned} \tag{3.15}$$

3.5 Energy-Momentum Tensor

If we follow the approach of relativistic electrodynamics, we may define the tensor {*T*} in terms of the field tensor {*F*} according to

$$T_{jk} \equiv \left(\frac{1}{4\pi}\right)\left[F_{j\lambda}F^{\lambda k} + \frac{1}{4}\delta_{jk}F_{st}F_{st}\right]$$

Using the field tensor to calculate the components of the energy-momentum tensor we find that the components are given by

$$T_{0\alpha} = \frac{-1}{4\pi}[(\overline{E} \, x \, \overline{B})_\alpha + V_4 V_\alpha] \ , \ \alpha = 1, 2, 3 \ ,$$

$$T_{00} = \frac{1}{8\pi}[E^2 + B^2 + V_4^2 + V^2] \ ,$$

$$T_{04} = \frac{i}{4\pi}[\overline{E} \bullet \overline{V}] \ ,$$

$$T_{4\alpha} = \frac{1}{4\pi}[V_4 E_\alpha + (\overline{V} \, x \, \overline{B})_\alpha] \ , \ \alpha = 1, 2, 3 \ ,$$

$$T_{44} = \frac{1}{8\pi}[V_4^2 + B^2 - E^2 - V^2] \ ,$$

and

$$T_{\alpha\beta} = \frac{1}{4\pi}\{E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta - \frac{1}{2}\delta\alpha\beta \, [E^2 + B^2 + V_4^2 - V^2]\} \ ,$$

where

$$\alpha, \beta = 1, 2, 3.$$

Equations (3.15) form a set of eight Maxwell-type equations which obviously reduce to Maxwell's four equations.

The wave equations for the new field quantities may be derived using standard assumptions.

$$\frac{\partial}{\partial t}(\overline{\Delta} \bullet \overline{V}) + \frac{1}{c}\frac{\partial^2 V_4}{\partial t^2} = \frac{4\pi}{c}\frac{\partial J_4}{dt} = \overline{\Delta} \bullet \frac{\partial \overline{V}}{dt} + \frac{1}{c}\frac{\partial^2 V_4}{\partial t^2}$$

and

$$\overline{\Delta} \bullet (\overline{\Delta} V_4) + \frac{1}{c}\overline{\Delta} \bullet \frac{\partial \overline{V}}{\partial t} = -a_0 \overline{\Delta} \bullet \frac{\partial \overline{E}}{\partial \gamma} = \frac{1}{c}\overline{\Delta} \bullet \frac{\partial \overline{V}}{dt} + \overline{\Delta} \bullet \overline{\Delta} V \ .$$

Therefore,

$$\Delta^2 V_4 - \frac{1}{c^2}\frac{\partial^2 V_4}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial J_4}{\partial t} - a_0 \overline{\Delta} \bullet \frac{\partial \overline{E}}{\partial \gamma} \ .$$

For the vector field we have:

$$\overline{\Delta}(\overline{\Delta} \bullet \overline{V}) + \frac{\overline{\Delta}}{c}\frac{\partial V_4}{\partial t} = -\overline{\Delta}(\frac{4\pi}{c}J_4)$$

and

$$\overline{\Delta} x (\overline{\Delta} x \overline{V}) + \Delta^2 \overline{V} + \frac{1}{c}\frac{\partial}{\partial t}\overline{\Delta} V_4 = \frac{4\pi}{c}\overline{\Delta} J_4 \ ;$$

therefore

$$\Delta^2 \overline{V} - \frac{1}{c^2}\frac{\partial^2 \overline{V}}{\partial t^2} = \frac{4\pi}{c}\overline{\Delta} J_4 + \frac{a_0}{c}\frac{\partial^2 \overline{E}}{\partial_t \partial\gamma} + a_0 (\overline{\Delta} x \frac{\partial \overline{B}}{\partial\gamma}) \quad .$$

But $\overline{\Delta} \bullet \overline{E} = 4\pi\rho - a_0 \frac{\partial V_4}{\partial\gamma}$ 70, so that

$$\Delta^2 V_4 - \frac{1}{c^2}\frac{\partial^2 V_4}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial J_4}{\partial t} - a_0 \frac{\partial}{\partial\gamma} 4\pi\rho - a_0 \frac{\partial V_4}{\partial t}$$

and $\overline{\Delta} x \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} \frac{4\pi}{c} \overline{J} - a_0 \frac{\partial \overline{V}}{\partial\gamma}$ 72 , so that

$$\Delta^2 \overline{V} - \frac{1}{c^2}\frac{\partial^2 \overline{V}}{\partial t^2} = \frac{4\pi}{c}\overline{\Delta} J_4 + a_0 \frac{\partial}{\partial\gamma}\frac{4\pi}{c}\overline{J} + 2\frac{\partial \overline{E}}{\partial t} - a_0 \frac{\partial \overline{V}}{\partial\gamma} .$$

Now the wave equations for the usual vector and scalar potentials are

$$\Delta^2 \overline{A} - \frac{1}{c^2}\frac{\partial^2 \overline{A}}{\partial t^2} = -\frac{4\pi}{c}\overline{J}$$

and

$$\Delta^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \quad .$$

We may differentiate these with respect to the mass density and substitute them into our wave equations and get

$$\Delta^2 V_4 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} V_4 = \frac{4\pi}{c^2}\frac{\partial J_4}{\partial t} + a_0^2 \frac{\partial^2 V_4}{\partial \gamma^2}$$

and

$$\Delta^2 \overline{V} - \frac{1}{c^2}\frac{\partial^2 \overline{V}}{\partial t^2} = \frac{4\pi}{c}\overline{\Delta} J_4 + a_0 \frac{\partial}{\partial\gamma}\left[2\frac{\partial \overline{E}}{\partial t} - a_0 \frac{\partial \overline{V}}{\partial\gamma} \right] \quad ,$$

where

$$V_4 \equiv V_4 + a_0 \frac{\partial\phi}{\partial\gamma} \quad and \quad \overline{V} = \overline{V} - a_0 \frac{\partial \overline{A}}{\partial\gamma} \quad .$$

The field energy density may be defined by

$$\xi \equiv \frac{1}{8\pi}[\, \overline{E} \bullet \overline{E} + \overline{B} \bullet \overline{B} + \overline{V} \bullet \overline{V} + V_4^2 \,] \quad ,$$

and the electrical Poynting vector may be defined by

$$\bar{S}_E \equiv \frac{c}{4\pi}(\bar{E}\, x\, \bar{B}) \ \ .$$

Now the electrical Poynting vector represents the outward flow of the electromagnetic field energy through a surface. If we take the total vector, whose components are $T_{0\alpha}$, to be the total flow of energy, then the vector with components $\frac{c}{4\pi}V_4 V_\alpha$ 81must be the outward flow of energy due to changes of the mass density within the surface. Let us designate the mass energy vector as

$$\bar{S}_m \equiv \frac{c}{4\pi}(V_4 \bar{V}) \ \ ,$$

so that the total energy vector is

$$\bar{S} = \bar{S}_E + \bar{S}_m$$

whose components are

$$S_\alpha = \frac{c}{4\pi}[\,(\bar{E}\, x\, \bar{B}\,)_\alpha + V_4 V_\alpha\,] = -(\frac{c}{i})T_{0\alpha} \ \ .$$

The Dynamic stress tensor may be defined as the three-dimensional tensor whose elements are

$$T^D_{\alpha\beta} = \frac{1}{4\pi}\{\,E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta - \frac{1}{2}\delta_{\alpha\beta}[\xi - 2V^2]\}$$

The Maxwell stress tensor is defined in electrodynamics as the three-dimensional tensor with elements

$$T^M_{\alpha\beta} = \frac{1}{4\pi}\{\,E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}[\,E^2 + B^2]\}$$

In terms of the Maxwell stress tensor, the Dynamic stress tensor may be written as

$$T^D_{\alpha\beta} = T^M_{\alpha\beta} - \{V_\alpha V_\beta - \frac{1}{2}\delta_{\alpha\beta}[V^2 - V_4^2]\} \ \ .$$

Then in terms of the above defined quantities

$$\{T\} = \begin{pmatrix} \xi & -\frac{1}{c}\overline{S} & \frac{i}{4\pi}(\overline{E}\bullet\overline{V}) \\ -\frac{i}{c}\overline{S} & \{T^D\} & \frac{i}{4\pi}V_4\overline{E}+(\overline{V}x\overline{B})] \\ \frac{i}{4\pi}(\overline{E}\bullet\overline{V}) & [V_4\overline{E}+(\overline{V}x\overline{B})] & \frac{1}{8\pi}[V_4^2+B^2-E^2-V^2] \end{pmatrix} .$$

Suppose we calculate the trace of the energy-momentum tensor:

$$t_R\{T\} = T_{jj} = T_{jj}^D + \xi + \frac{1}{8\pi}[V_4^2+B^2-E^2-V^2]$$

$$= \frac{1}{4\pi}[B^2+V_4^2] + T_{\alpha\alpha}^4$$

$$= \frac{1}{4\pi}[B^2+V_4^2] + \frac{1}{4\pi}[E^2+B^2-V^2-\frac{3}{2}(E^2+B^2+V_4^2-V^2)]$$

$$= \frac{1}{4\pi}[\frac{1}{2}B^2-\frac{1}{2}V^2-\frac{1}{2}V_4^2]$$

$$= \frac{1}{8\pi}[B^2+V^2-E^2-V_4^2] \quad .$$

3.6 Force Density Vector.

The force density vector may be defined in terms of the divergence of the energy-momentum tensor. Therefore, suppose we calculate the five-dimensional divergence of the tensor $\{T\}$, or

$$\frac{\partial T_{jk}}{\partial x^k} = \frac{1}{4\pi}\frac{\partial}{\partial x^k}[F_{j\lambda}F_{\lambda k} + \frac{1}{4}\delta_{jk}F_{st}F_{st}] \quad .$$

Because of the antisymmetry of F_{jk}, the first term may be written as

$$\frac{\partial F_{j\lambda}}{\partial x^k}F_{\lambda k} = \frac{\partial F_{\lambda j}}{\partial x^k}F_{k\lambda} \quad .$$

By interchanging the indices k and l

$$\frac{\partial F_{j\lambda}}{\partial x^k}F_{\lambda k} = \frac{\partial F_{kj}}{\partial x^\lambda}F_{\lambda k} = \frac{1}{2}\left(\frac{\partial F_{j\lambda}}{\partial x^k} + \frac{\partial F_{j\lambda}}{\partial x^\lambda}\right)F_{\lambda k} \quad .$$

Using the Bianchi identity

$$\frac{\partial F_{j\lambda}}{\partial x^k} + \frac{\partial F_{kj}}{\partial x^\lambda} + \frac{\partial F_{\lambda k}}{\partial x^j} = 0 \quad ,$$

the terms contained within the parentheses may be written as

$$\frac{\partial F_{j\lambda}}{\partial x^k}F_{\lambda k}=\frac{1}{2}\frac{\partial F_{\lambda k}}{\partial x^j}F_{\lambda k}=-\frac{1}{4}\frac{\partial(F_{\lambda k}F_{\lambda k})}{\partial x^i} \quad .$$

Substituting this back into the expression for the divergence, the last term will be canceled because *l*, k, s, and t are dummy indices. Then the divergence becomes

$$\frac{\partial T_{jk}}{\partial x^k}=\frac{1}{4\pi}F_{j\lambda}\frac{\partial F_{\lambda k}}{\partial x^k} \quad .$$

By interchanging the indices k and *l* on the right-hand side we obtain

$$\frac{\partial T_{jk}}{\partial x_k}=\frac{1}{4\pi}F_{jk}\frac{\partial F_{k\lambda}}{\partial x^\lambda} \quad . \tag{3.16}$$

The Dynamic force density five-vector may now be defined as

$$K \equiv Div_{\ 5}\{T\} \quad .$$

Therefore, the components of K are given by

$$K_j=\frac{1}{4\pi}F_{jk}\frac{\partial F_{k\lambda}}{\partial x^\lambda} \quad .$$

But the five-vector current density is given by

$$\frac{\partial F_{k\lambda}}{\partial x^\lambda}=\frac{-4\pi}{c}J_k \quad .$$

The components of the five-vector force density become

$$K_j=\frac{-1}{4\pi}F_{kj}\left(\frac{4\pi}{c}J_k\right)=\frac{-1}{c}J_kF_{kj} \quad .$$

Now, since J_k = (icρ, J, J_4), then

$$\begin{aligned} K_0&=\frac{i}{c}[\bar{J}\bullet\bar{E}+J_4V_4] \quad , \\ \bar{K}&=\rho[\bar{E}+\frac{1}{c}(\bar{u}\ x\ \bar{B})]+\frac{J_4}{c}\bar{V} \end{aligned} \quad , \tag{3.17}$$

and

$$K_4=\rho V_4-\frac{\bar{J}\bullet\bar{V}}{c} \quad ,$$

where J = ρu. These then are the components of the force density five-vector resulting from a gauge field in the Dynamic Theory. These components reduce to the four components of the Lorentz force density should V_4 = V = 0.

With the interpretation that the four force density components with subscript 1 through 4 are the force density vectors which appear in the First Law as F_α, then the force density vector provides the connection between the First Law and the geometry of the sigma manifold discussed in section 2.9. Thus, the existence of the vector field φ_i is also demanded by the Dynamic Theory and need not exist as a separate assumption.

3.7 Equation of Energy Flow.

Consider the zeroth component of the Dynamic force density five-vector

$$K_0 = \frac{\partial T_{0k}}{\partial x^k} = \frac{\partial T_{00}}{\partial x^0} + \frac{\partial T_{0\alpha}}{\partial x^\alpha} + \frac{\partial T_{04}}{\partial x^4} \quad .$$

Then

$$\frac{i}{c}[\overline{J} \bullet \overline{E} + J_4 V_4] = \frac{\partial \xi}{\partial (ict)} - \frac{i}{c}\frac{\partial S_\alpha}{\partial x^\alpha} + \frac{i}{4\pi}\frac{\partial (\overline{E} \bullet \overline{V})}{\partial x^4}$$

or

$$\frac{1}{c}[\overline{J} \bullet \overline{E} + J_4 V_4] = \frac{1}{c}\frac{\partial \xi}{\partial t} + \frac{1}{c}\frac{\partial S_\alpha}{\partial x^\alpha} - \frac{1}{4\pi}\frac{\partial (\overline{E} \bullet \overline{V})}{\partial x^4} \quad ,$$

or, since $x^4 = \gamma/a_0$,

$$-\overline{J} \bullet \overline{E} - J_4 V_4 = \frac{\partial \xi}{\partial t} + \frac{\partial S_\alpha}{\partial x^\alpha} - \frac{a_0 c}{4\pi}\frac{\partial (\overline{E} \bullet \overline{V})}{\partial \gamma} \quad .$$

Rearranging the terms

$$div\ \overline{S} + \frac{\partial \xi}{\partial t} = -\overline{J} \bullet \overline{E} - J_4 V_4 + \frac{a_0 c}{4\pi}\frac{\partial (\overline{E} \bullet \overline{V})}{\partial \gamma}$$

and separating out the electrical Poynting vector leads to

$$div\ \overline{S}_E + \frac{\partial \xi}{\partial t} = -\overline{J} \bullet \overline{E} - J_4 V_4 - div\ \overline{S}_m + \frac{a_0 c}{4\pi}\frac{\partial (\overline{E} \bullet \overline{V})}{\partial \gamma} \quad .$$

This then is the five-dimensional energy flow equation.

3.8 Momentum Conservation

The expression for the conservation of momentum may be obtained from the space portion of the force density five-vector

$$K = \frac{\partial T_{\alpha k}}{\partial x^{k}} = \partial \frac{T_{\alpha 0}}{\partial x^{0}} + \frac{\partial T_{\alpha\beta}}{\partial x^{\beta}} + \frac{\partial T_{\alpha 4}}{\partial x^{4}} \, , \; \alpha, \beta = 1, 2, 3 \, .$$

But $\frac{\partial T_{\alpha\beta}}{\partial x^{\beta}}$ 110 is the three-dimensional divergence of the Dynamic stress tensor $\{T^{D}\}$, therefore,

$$\overline{K} = -\frac{1}{c^{2}} \frac{\partial \overline{S}}{\partial t} + div \; \{T^{D}\} + \frac{a_{0}}{4\pi} \frac{\partial}{\partial \gamma} [V_{4} \overline{E} + (\overline{V} \; x \; \overline{B})] \quad .$$

If we consider a volume in which all the material is contained and outside of which the field vanishes, then integrating over this volume yields

$$\int_{v} \{ \overline{K} + \frac{1}{c^{2}} \frac{\partial \overline{S}}{\partial t} - \frac{a_{0}}{4\pi} \frac{\partial}{\partial \gamma} [V_{4} \overline{E} + (\overline{V} x \overline{B})]\} \; dv = \int_{v} div \{T^{D}\} \, dv \quad .$$

The integral of K gives the total force (i.e., the time derivative of the mechanical momentum p less the vector $\left(\gamma\dot{v}^{\alpha} / \sqrt{1-\beta^{2}}\right)$ 113. Now define the vector

$$\overline{g} \equiv \frac{\overline{S}}{c^{2}} - \frac{a_{0}}{4\pi} \int \{ \frac{\partial}{\partial \gamma} [V_{0} \overline{E} + (\overline{V} \; x \; \overline{B}) + \frac{\gamma\dot{v}^{\alpha}}{\sqrt{1-\beta^{2}}} \} \, dt \quad .$$

Then define

$$\int_{v} \overline{g} \, d\sigma \equiv \overline{G} \quad ,$$

so that

$$\frac{d}{dt} (\overline{p} + \overline{G}) = \int_{v} div \{T^{D}\} \, dv \quad .$$

Using the divergence theorem the volume integral may be converted to a surface integral so that

$$\frac{d}{dt} (\overline{p} + \overline{G}) = \int_{s} \{T^{D}\} \bullet \overline{n} \, da \quad .$$

If the field vanishes outside of V, it must do so also on the boundary surface s, hence

$$\frac{d}{dt} (\overline{p} + \overline{G}) = 0 \quad .$$

Therefore, it is not the mechanical momentum p but the quantity p + G which is conserved. Therefore, we must interpret G as the momentum of the field and

$$\bar{g} = \frac{\bar{S}}{c^2} - \frac{a_0}{4\pi}\int\left(\frac{\partial}{\partial\gamma}\left[\bar{V}\bullet\bar{E} + (\bar{V}x\bar{B})\right] + \frac{4\pi\gamma\dot{\bar{v}}}{\sqrt{1-\beta^2}}\right)dt$$

as the momentum density of the field.

3.9 Gauge Field Pressure

The Dynamic stress tensor is given by

$$T^D_{\alpha\beta} = \frac{1}{4\pi}\{E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta - \frac{1}{2}\delta_{\alpha\beta}\, E^2 + B^2 + V_4^2 - V^2]\}$$
$$= -\frac{1}{8\pi}[E^2 + B^2 - V^2] - \frac{3}{8\pi}V_4^2 \quad .$$

Now separate the three-dimensional dynamic stress tensor into a traceless and an isotropic tensor.

$$T^D_{\alpha\beta} = \frac{1}{4\pi}\{E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta - \frac{1}{2}\gamma_{\alpha\beta}[E^2 + B^2 + V_4^2 - V^2\}]$$
$$= \frac{1}{4\pi}\{E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta - \frac{1}{8\pi}\delta_{\alpha\beta}[E^2 + B^2 + V_4^2 - V^2]\}$$
$$= \frac{1}{4\pi}[E_\alpha E_\beta + B_\alpha B_\beta - V_\alpha V_\beta] - (\frac{2}{3})(\frac{1}{8\pi})\delta_{\alpha\beta}[E^2 + B^2 - V^2]$$
$$-(\frac{1}{3})(\frac{1}{8\pi})\delta_{\alpha\beta}[E^2 + B^2 + 3V_4^2 - V^2]$$
$$\equiv t_{\alpha\beta} + \tau'_{\alpha\beta}$$

where

$$t_{\alpha\beta} \equiv \frac{1}{4\pi}\{E_\alpha E_\beta + B_\alpha B_\beta + V_\alpha V_\beta - (\frac{1}{3\pi})\delta_{\alpha\beta}[E^2 + B^2 - V^2]$$

and

$$\tau'_{\alpha\beta} \equiv -(\frac{1}{24\pi})\delta_{\alpha\beta}[E^2 + B^2 + 3V_4^2 - V^2] \quad .$$

Now

$$t_r\{t_{\alpha\beta}\} = (\frac{1}{4\pi})[E^2 + B^2 - V^2 - (E^2 + B^2 - V^2)] \equiv 0$$

and

$$t_r\{\tau'_{\alpha\beta}\} = -(\frac{1}{8\pi})[E^2 + B^2 + 3V_4^2 - V^2] \quad .$$

Consider the definition

$$\frac{1}{3} t \delta_{\alpha\beta} \equiv \tau' = -(\frac{1}{24\pi}) \delta_{\alpha\beta} [E^2 + B^2 + 3V_4^2 - V^2] \quad .$$

Then

$$t = -(\frac{1}{8\pi}) [E^2 + B^2 + 3V_4^2 - V^2]$$

and

$$\tau'_{\alpha\beta} = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$$

The isotropic part of the stress tensor is usually called the "pressure." Therefore, define 3p = t in accordance with customary notation, so that

$$p = -(\frac{1}{24\pi}) [E^2 + B^2 + 3V_4^2 - V^2] \quad .$$

With the exception of the factor of 3 this reduces to the "radiation pressure" for an electromagnetic field when V = V_0 =0. Note that this pressure may be zero since it is the sum and difference of squares, or p = 0, when

$$V^2 = E^2 + B^2 + 3V_4^2 \quad .$$

This may prove to be an important point when considering boundary conditions in cosmology or the study of elementary particles.