Chapter 4. QUANTIZATION IN FIVE DIMENSIONS

The preceding development provides a tremendous wealth of mathematical abstractions. However, there seems within it no readily apparent method of interpreting the new fields. If there appears to be no physical entity which may be associated with the new field quantities, then the development will have gone for naught. On the other hand, with the notion of nuclear fields in mind it seems that if the new field quantities are included in a quantized picture, then perhaps the relation to nuclear fields may be made.

In the following the requirement for quantization is provided by appropriate restrictions upon a system whose description is taken from the Dynamic Theory. However, the use of the five-dimensional Dirac equation has not yet been shown to result from the Dynamic Theory. Schrodinger's quantum mechanics may be obtained using London's work, but I am not aware of a procedure to arrive logically at Dirac's equation even though I feel that the method exists. As it now stands, the use of the generalized Dirac equation must be accepted as an independent fundamental assumption.

4.1 Quantization.

The system under consideration now is a five-dimensional system with arc element

$$(dq^0)^2 = f(d\sigma)^2 \quad .$$

Now since our system is an E-conservative, dE = 0, system the principle of increasing entropy requires that $(dq^0)^2 \geq 0$ so that $f(d\sigma)^2 \geq 0$. Introducing the quantization conditions results in

$$\int \phi_j \, dx^j = 2\pi i n \, , \; j = 0, 1, 2, 3, 4 \quad ,$$

where

$$\phi_j \equiv \pm \frac{\partial \ell n f^{1/2}}{\partial x^j} \text{ and } x^0 \equiv ct \; , \; x^1 \equiv q^1, x^2 \equiv q^2, x^3, x^4 \equiv \frac{\gamma}{a_0} \quad .$$

If we restrict ourselves to a $(d\sigma)^2$ space which is the local Euclidean space, then $(d\sigma)^2$ is the five-dimensional Minkowski-type manifold; using London's work we would produce a five-dimensional quantum dynamical system.

4.2 Five-Dimensional Hamiltonian.

We previously showed that the principle of increasing entropy resulted in

$$\delta \int \gamma \sqrt{(dq^0)^2} = 0$$

as the variational principle for a local Euclidean manifold. Since multiplication by a constant does not change the problem we may take our variational problem to be

$$\delta \int \gamma \, c^2 \sqrt{(dq^0)^2} = 0 \quad .$$

Defining the velocity vector as $u^j = dx^j/dq^0$ and the momentum as $p_j = \partial L/\partial u^j = \gamma g_{jk} u^k$, where we have used the fact that $g_{jk}u^j u^k = 1$, then we may form the contravariant momentum as

$$p_j = g^{jk} p_k = g^{jk} \gamma \, g_{k\ell} u^\ell \quad ,$$

so that

$$\begin{aligned} p_j p^j &= (\gamma \, g_{j\ell} u^j)(g^{jk} \gamma \, g_{k\ell} u^\ell) = \gamma^2 \delta_{j\ell} u^\ell g_{jk} u^k \\ &= \gamma (\gamma \, g_{ik} u^j u^k) \\ &= \gamma^2 c^2 \quad , \end{aligned} \tag{4.1}$$

since $\gamma c^2 = \gamma g_{jk} u^j u^k$. Equation (4.1) is the five-dimensional "momentum-energy" equation.

We may now follow London's procedure to obtain our wave function for the five-dimensional system. However, a quicker way to investigate the effect of the Dynamic Theory upon quantum mechanics would seem to be that of adopting Dirac's equation in a five-dimensional form and following a development analogous to standard four-dimensional relativistic quantum mechanics. With this in mind, then we shall adopt the form

$$h = i\left(\tilde{\alpha}_1 \frac{\partial}{\partial x^1} + \tilde{\alpha}_2 \frac{\partial}{\partial x^2} + \tilde{\alpha}_3 \frac{\partial}{\partial x^3} + \tilde{\alpha}_4 \frac{\partial}{\partial x^4}\right) - \tilde{\beta} \tag{4.2}$$

to be the five-dimensional specific Hamiltonian operator. The partial derivative operators are specific operators and hence are dimensionless in natural units. In Eqn. (4.1) the α's and β do not involve derivatives and must be Hermitian in order that h be Hermitian.

By taking the four partial derivatives in Eqn. (4.1) as the four-vector specific momentum operator we may write

$$h = -(\tilde{\alpha} \bullet \bar{p} + \tilde{\beta}) \quad . \tag{4.3}$$

4.3 Five-Dimensional Dirac Equation.

If we take $p^0|\rangle = h|\rangle$ 10 and require that the α's and β are chosen such that solutions of this equation are also solutions of Eqn. (4.3), we find the restrictions imposed upon the choice of the α's and β to be:

$$(\tilde{\alpha} \bullet \overline{p}) = p^2 \quad ,$$
$$\tilde{\beta}^2 = 1 \quad ,$$

and

$$\tilde{\alpha}\tilde{\beta} + \tilde{\beta}\tilde{\alpha} = 0 \quad , \qquad (4.4)$$

where natural units, c = 1, are used.

A set of 8 x 8 matrices satisfying the requirements of Eqn. (4.4) is

$$\tilde{\beta} = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad \tilde{\alpha}_i = \begin{pmatrix} 0 & \alpha_i \\ \alpha_i & 0 \end{pmatrix} \; i = 1, 2, 3, \quad , \tilde{\alpha}_4 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad , \qquad (4.5)$$

where

$$\beta = \begin{pmatrix} I & O \\ 0 & -I \end{pmatrix} \; , \; \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} i = 1, 2, 3 \quad , and \;\; A = \begin{pmatrix} 0 & I \\ I & O \end{pmatrix}$$

and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, and \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Then the five-dimensional Dirac equation may be taken to be

$$i \frac{\partial}{\partial t} \Psi(x) = (i\,\tilde{\alpha} \bullet \Delta - \tilde{\beta})\,\Psi(x) \quad , \qquad (4.6)$$

where the Δ 17is a four-dimensional operator. By defining

$$\gamma^0 \equiv \tilde{\beta} \;\; ; \;\; \gamma^\mu \equiv - \tilde{\beta}\,\tilde{\alpha}_\mu \;\; (\mu = 1, 2, 3, 4) \quad ,$$

then Eqn. (4.6) may be written as

$$(i\,\partial_j \gamma^j + 1)\,\Psi(x) = 0 \quad .$$

By virtue of the properties of the $\tilde{\alpha}$'s 20 and $\tilde{\beta}$ 21 plus the fact that

$$g^{jk} = \begin{cases} 1 \; for \; j = k = 0 \\ -1 \; for \; j = k = 1, 2, 3, 4 \\ 0 \; for \; j \neq k \end{cases} \quad ,$$

the anticommutator of the γ-matrices must satisfy

$$\{\gamma^j, \gamma^i\} = 2\, g^{ji} \quad .$$

4.4 "Lorentz" Covariance.

Under a five-dimensional Lorentz transformation

$$x'^j = L_{kj}\, x^k$$

we shall suppose each component of the wave function Ψ(x) transforms into a linear combination of all four components:

$$\Psi\,(x)\, \vec{L} T\, \Psi'\,(x') = S\, \Psi\,(x) \quad ,$$

where S is a Dirac spinor satisfying

$$S^{-1}\, \gamma^j\, S = L^j_k\, \gamma^k \quad .$$

(4.7)

By using an infinitesimal Lorentz transformation given by

$$L^j_k = g^j_k + d\theta\, \varepsilon^j_k \tag{4.8}$$

where ε^j_k 28 are a set of 16 numbers, then S (θ) may be shown to be given by

$$S\,(\theta) = exp\,(T \int_0^\theta d\theta) \quad , \tag{4.9}$$

where the matrix T is given by

$$T = \frac{1}{4}\, \varepsilon_{jk}\, \gamma^j\, \gamma^k \quad .$$

Equations (4.7), (4.8), and (4.9) suffice to guarantee the Lorentz covariance of the five-dimensional Dirac equation.

Following standard quantum mechanical procedure we shall adopt the probability current density to be

$$j^k\,(x) = \overline{\psi}\,(x)\, \gamma_k\, \psi\,(x)$$

with the requirements that:

$\partial_k\, j^i = 0$,
j^ktransforms as a contravariant vector, and
j^k must be real.

4.5 Spin.

In the three-dimensional space the angular momentum is given by the vector L as the cross product of the coordinates and momenta. We shall then define the angular four-momentum to be the four-dimensional cross product

$$\bar{L} \equiv \varepsilon_{ijk} x^j p^k$$

where x^4 is the mass density and

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ if any two indices are alike,} \\ 1 \text{ for even permutation to align indices in ascending order,} \\ -1 \text{ for odd permutation to align indices in ascending order.} \end{cases}$$

Then the commutator of the components of the angular four-momentum with the specific Hamiltonian is not zero; for instance

$$[L_3,h] = i\gamma^0\gamma^1 p^2 - i\gamma^0\gamma^2 p^1 + i\gamma^0\gamma^4 p^1 - i\gamma^0\gamma^1 p^4 + i\gamma^0\gamma^4 p^2 - i\gamma^0\gamma^2 p^4 \quad .$$

Now suppose there exists a four-spin vector S such that the sum of the angular four-momentum and the four-spin vector commutes with the specific Hamiltonian; then if we define a new three-spin vector u given by the components $u_i = i\gamma^4\gamma^1$, $u_2=i\gamma^4\gamma^2/2$, and $u_3=i\gamma^4\gamma^3/2$, and take the usual spin vector s, given by $s_1=i\gamma^2\gamma^3/2$, $s_2=i\gamma^1\gamma^{3/2}$, and $s_3=i\gamma^1\gamma^2/2$, the components of the four-spin vector may be shown to be

$$S_1 = s_1 - u_2 - u_3 \quad ,$$
$$S_2 = s_2 + u_1 - u_3 \quad ,$$
$$S_3 = s_3 + u_1 + u_2 \quad ,$$

and

$$S_4 = s_1 - s_2 + s_3 \quad .$$

In analogy with standard relativistic quantum mechanics the eigenvalues of the four-spin components can be shown to be $\pm\sqrt{\frac{3}{4}}$ 36. It may also be shown that the set of observables P, h, and S where P is the four-momentum and S is the four-spin, form a complete set of commuting observables.

4.6. Dirac Equation with Fields.

In analogy with relativistic quantum mechanics we take the five-dimensional Dirac equation to be

$$[(i\partial_j - \phi_j)\gamma^j + 1]\,\Psi = 0 \quad , \tag{4.10}$$

where φ_j is five-vector potential. By operating on the left with $[(i\partial_j - \phi_j)\gamma^i - 1]$ 38 and separating $\gamma^j\gamma^k$ 39 into symmetric and antisymmetric parts as

$$\gamma^i\gamma^k = \frac{1}{2}\{\gamma^i,\gamma^k\} + \frac{1}{2}[\gamma^j,\gamma^k] \equiv g^{ik} + \sigma^{jk} \quad ,$$

then Eqn. (4.10) becomes

$$[(i\partial_j - \phi_j)(i\partial^j - \phi^j) - 1 + (-\partial_j\partial_k + \phi_j\phi_k - i\phi_j\partial_k - i\partial_j\phi_k)\,\sigma^{jk}]\Psi = 0. \tag{4.11}$$

Separating $\partial_j\phi_k$ 42 into symmetric and antisymmetric parts as

$$\partial_j\phi_k = \frac{1}{2}(\partial_j\phi_k) + \frac{1}{2}(\partial_j\phi_k - \partial_k\phi_j)$$

and defining the field tensor as

$$F_{jk} = \partial_j\phi_k - \partial_k\phi_j \quad ,$$

Eqn. (4.11) becomes

$$[(i\partial_j - \phi_j)(i\partial^k - \phi^k) - 1 - \frac{1}{2}iF_{jk}\sigma^{jk}]\,\Psi = 0 \quad . \tag{3.28}$$

Now since

$$\sigma^{jk} = \begin{vmatrix} 0 & \dot{x}^1 & \dot{x}^2 & \dot{x}^3 & \dot{x}^4 \\ -\dot{x}^1 & 0 & -2is^1 & 2is^2 & n^1 \\ -\dot{x}^2 & 2is^1 & 0 & -2is^3 & n^2 \\ -\dot{x}^3 & -2is^2 & 2is^3 & 0 & n^3 \\ -\dot{x}^4 & -n^1 & -n^2 & -n^3 & 0 \end{vmatrix} \quad ,$$

where

$$\sigma^j 4 \equiv n^j = 1, 2, 3, \text{and}$$

$$F_{jk} = \begin{vmatrix} 0 & E_1 & E_2 & E_3 & V_0 \\ -E_1 & 0 & B_3 & B_2 & V_1 \\ -E_2 & B_3 & 0 & -B_1 & V_2 \\ E_3 & -B_2 & B_1 & 0 & V_3 \\ -V_0 & -V_1 & -V_2 & -V_3 & 0 \end{vmatrix}$$

plus recalling the seven Maxwell-type equations from Eqn. (3.15)

$$\overline{\Delta} V_4 + \frac{1}{c}\frac{\partial \overline{V}}{\partial t} + a_0 \frac{\partial \overline{E}}{\partial \gamma} = 0, \quad \overline{\Delta} \bullet \overline{V} + \frac{\partial V_4}{\partial t} = -\frac{4\pi}{c} J_4,$$

$$\overline{\Delta}\, x\, \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} = \frac{4\pi \overline{J}}{c} - a_0 \frac{\partial \overline{V}}{\partial \gamma}, \qquad \overline{\Delta}\, x\, \overline{V} + a_0 \frac{\partial \overline{B}}{\partial \gamma} = 0,$$

$$\overline{\Delta} \bullet \overline{B} = 0, \qquad \overline{\Delta}\, x\, \overline{E} + \frac{1}{c}\frac{\partial \overline{B}}{\partial t} = 0, \qquad \overline{\Delta} \bullet \overline{E} = 4\pi\rho - a_0 \frac{\partial V_4}{\partial \gamma}, \tag{4.13}$$

then Eqn. (4.12) may be written as

$$[(i\partial_j - \phi_j)(i\partial^k - \phi^k) - 1 + 2\overline{B} \bullet \overline{s} - i\overline{E} \bullet \overline{x} - iV_4 \dot{x}^4 - i\overline{n} \bullet \overline{V}]\, \Psi = 0.$$

and thus becomes the Dirac equation with fields E, B, V_4, and V.

Suppose we consider a system without an electric charge so that p = J = 0, then by Eqn. (4.13) we still have

$$\overline{\Delta} \bullet \overline{E} = -a_0 \frac{\partial V_4}{\partial \gamma} \qquad and \qquad \overline{\Delta}\, x\, \overline{B} - \frac{1}{c}\frac{\partial \overline{E}}{\partial t} = -a_0 \frac{\partial \overline{V}}{\partial \gamma}$$

and, therefore, there will still be a magnetic moment.

4.7. Allowed Fundamental Spin States.

In the five-dimensional quantization of the space-time-mass manifold three spin vectors appear. One of these is the familiar three-component spin vector of relativistic quantum mechanics. The second of the three is a new three-component spin vector while the remaining one is a four-component spin vector.

Using the theorem:

> If α satisfies $\alpha^2 = a^2$ where a is a number, then the eigenvalues of α are $\pm a$.

Then it is not difficult to show that the component eigenvalues are

$$s_\alpha = \pm \frac{1}{2}, u_\alpha = \pm \frac{1}{2}, S_j^2 = \frac{3}{4}, \alpha = 1,2,3 \ and \ j = 1,2,3,4 \ .$$

If, in analogy with the eigenvalues for the total angular momentum, we write

$$S_j^2 = \frac{3}{4} = S_j(S_j + 1) \ ,$$

then the possible eigenvalues becomes

$$s_\alpha = +_- \frac{1}{2}, u_\alpha = +_- \frac{1}{2}, S_j = \frac{1}{2}, -\frac{3}{2} \ .$$

However, the following relations, which were shown to be required for S to commute with the specific Hamiltonian, restrict the number of possible combinations of these eigenvalues.

$$S_1 = s_1 - u_2 - u_3 \ ,$$
$$S_2 = s_2 + u_1 - u_3 \ ,$$
$$S_3 = s_3 + u_1 + u_2 \ ,$$

and

$$S_4 = s_1 - s_2 + s_3 .$$

The question to be asked now is, how many combinations of the above eigenvalues are allowed?

For $S_1 = 1/2$ the combination $s_1 = -1/2$ and $u_2 = 1/2$ is impossible.
For $S_1 = 1/2$ the combination $s_2 = -1/2$ and $u_1 = -1/2$ is impossible.
For $S_3 = 1/2$ the combination $s_3 = -1/2$ and $u_1 = -1/2$ is impossible.
For $S_4 = 1/2$ the combination $s_1 = -1/2$ and $s_2 = 1/2$ is impossible.
For $S_1 = -3/2$ only one combination is possible: $s_1 = -1/2$, $u_2 = 1/2$, and $u_3 = 1/2$.
For $S_2 = -3/2$ only one combination is possible: $s_2 = -1/2$, $u_1 = -1/2$, and $u_3 = 1/2$.
For $S_3 = -3/2$ only one combination is possible: $s_3 = -1/2$, $u_1 = -1/2$, and $u_2 = -1/2$.
For $S_4 = -3/2$ only one combination is possible: $s_1 = -1/2$, $s_2 = 1/2$, and $s_3 = -1/2$.

Now because S_4 is a combination of the first terms of each of the components S_1, S_2, S_3, not all of the above listed 16 combinations are possible.

For $S_4 = 1/2$ the following combinations of (s_1, s_2, s_3, u_1, u_2, u_3) are possible.

(1) (2, 2, 2 ; - 2, 2, - 2 for $S_1 = S_2 = S_3 = 2$
(2) (2, 2, 2 ; 2, - 2, 2) for $S_1 = S_2 = S_3 = 2$
(3) (2, - 2, - 2 ; 2, 2, - 2) for $S_1 = S_2 = S_3 = 2$
(4) (- 2, - 2, 2 ; 2, - 2, - 2) for $S_1 = S_2 = S_3 = 2$

The remaining combinations are:

(5) (- 2, 2, - 2 ; - 2, - 2, - 2) for $S_4 = S_3 - \beta$; $S_1 = S_2 = 2$
(6) (-2, 2, - 2 ; 2, 2, 2) for $S_4 = S_1 - \beta$; $S_2 = S_3 = 2$
(7) (2, - 2, - 2 ; - 2 - 2, 2) for $S_2 = S_3 = - \beta$; $S_1 = S_4 = 2$
(8) (- 2, - 2, 2 ; - 2, 2, 2) for $S_1 = S_3 = - \beta$; $S_2 = S_4 = 2$.

Thus there is an octet of possible combinations. There are also some obvious symmetries in these combinations. An aid in seeing these symmetries is the vector defined as t where

$$t_1 \equiv -(u_2 + u_3)\ ;\ t_2 \equiv u_1 - u_3\ ;\ t_3 \equiv u_2 + u_1$$

Then for each of the eight combinations above we find (t_1, t_2, t_3) given by

(1) t= (0,0,0)
(2) t=(0,0,0)
(3) t= (0,1,1)
(4) t= (1,1,0)
(5) t= (1, 0, -1)
(6) t= (-1, 0, 1)
(7) t= (0, -1, -1)
(8) t= (-1, -1, 0)

Thus, the eight combinations correspond to four distinct t vectors which carry a ± sign. Or

t_1=(0, 0,0) ;t_2 = (0, 1, 1) ; t_3 = (1, 1, 0) ; t_4 = (1, 0, -1)

For + t_α we have:

$t_1 \rightarrow$ (s;u) = (2, 2, 2 ; - 2, 2, - 2)
$t_2 \rightarrow$(s;u) = (2, - 2, - 2 ; 2, 2, - 2)
$t_3 \rightarrow$ (s;u) = (- 2, - 2, 2 ; 2, - 2, - 2)
$t_4 \rightarrow$ (s;u) = (- 2, 2, - 2 ; - 2, - 2, - 2) .

For -t_α we have:

$-t_1 \rightarrow$ (s;u) = (s,*u*) = (2, 2, 2 ; 2, - 2, 2)
$-t_2 \rightarrow$ (s;u) = (s,*u*) = (2, - 2, - 2 ; - 2, - 2, 2)
$-t_3 \rightarrow$ (s;u) = (s,*u*) = (- 2, - 2, 2 ; - 2, 2, 2)
$-t_4 \rightarrow$ (s;a) = (s,*u*) = (- 2, 2, - 2 ; 2, 2, 2)

Now by defining the vectors:

a = (2, 2, 2) ; b= (- 2, 2, - 2)
c = (2, - 2, - 2) ; d =(2, 2, - 2)

We may write

$t_1 \rightarrow (s;u) = (a;b)$ $\quad$ $-t_1 \rightarrow (s;u) = (a;-b)$
$t_2 \rightarrow (s;u) = (c;d)$ $\quad$ $-t_2 \rightarrow (s;u) = (c;-d)$
$t_3 \rightarrow (s;u) = (-d;c)$ $\quad$ $-t_3 \rightarrow (s;u) = (-d;c)$
$t_4 \rightarrow (s;u) = (b;-a)$ $\quad$ $-t_4 \rightarrow (s;u) = (b;a)$

The octet is then made up of the combinations:

$(a;\pm b);\ (c;\pm d);\ (b;\pm a);\ (-d;\pm c)$.

The appearance of octets for basic quantum numbers is reminiscent of elementary particle theory. Thus, the Dynamic Theory seems to give promise to the hope of tying elementary particles to fundamental principles in a new way.

B. Quantized Fields

Much difficulty was encountered in trying to find a solution to the wave equations. This stimulated a return to thoughts of fundamental particles. The motivation for this change was primarily the feeling that it would be more productive to get away from the wave solutions for a while, but also there was the haunting feeling, retained for some five years, that the new fields played a role in particle structure. This feeling was based primarily on the role the new fields appear to play in the five-dimensional quantization and their role in the self-energy of charged particles.

4.8. Quantum Condition Applied to Particles.

The quantum condition

$$\int \phi_j \, dx^j = 2\pi i N, \quad j = 0, 1, \bullet\bullet\bullet, n, \tag{4.14}$$

was required when generalized isentropic states were considered. Given the thermodynamic basis for the three fundamental laws, it seems natural to think that if the Dynamic Theory were to say anything about fundamental particles then it should probably come from considering generalized isentropic states. Thus, the quantum condition, Eqn. (4.14), should play a crucial role. This also was the condition from which London began his work, which showed that this condition produces quantum mechanics, but quantum mechanics describes interactions between particles such as electrons and nuclei. It does not specify what types of particles are allowed.

That the three adopted laws must apply to individual fundamental particles is tantamount to the notion that these three laws must specify what particles are allowed and, thereby, must specify their allowed fields. If we again look at the quantum condition, Eqn. (4.14), we see that it is given as a line integral that must have a quantized value. Our usual first encounter with a line integral involves the evaluation of a given line integral when the path is specified. Because the quantum condition represents a line integral that may only have certain values, London asked a legitimate question when he asked what paths would be allowed given the electrostatic potential in advance[8]. This points out that there are three parts to any line integral, the integrand, the path, and the integral value.

Another question that may be asked of the quantum condition is given that the integral value may only be $2\pi iN$, what are the possible φ allowed for a particle that must retain its identity along any path? This is equivalent to asking what fields are fundamental particles allowed to have if we are free to move them anywhere in the manifold? To be more specific, we are asking what θ are allowed by the quantum condition if the dx^j are to be independent?

If the dx^j are to be independent, then we may choose all dx^j to be zero except dx^k. Then the quantum condition requires

$$\int \phi_k \, dx^k = 2\pi iN \quad (no\ sum\ on\ k). \tag{4.15}$$

Equation (4.15) must be true for all k, and because we are free to set the path, then the φ_k must reflect the quantization represented by the integer N. Therefore,

$$\phi_j = N_j \tilde{\phi}_j \quad (no\ sum) \quad , \tag{4.16}$$

where the may not be quantized. Thus, Eqn. (4.16) represents the first response of the quantum condition to the question concerning what φ_j are allowed for fundamental particles; the gauge potentials must be quantized.

The definition of the gauge potentials is

$$\phi_j = \frac{\partial \ln f^{\frac{1}{2}}}{\partial x_j} \quad , \tag{4.17}$$

where f was the gauge function. The field tensor was defined by

$$F_{jk} = \phi_{j,k} - \phi_{k,j} \quad , \tag{4.18}$$

where covariant differentiation is required. There are restrictions placed upon these fields, for they must obey the set of eight differential equations given by Eqn. (3.15).

4.9. Radial Field Dependence.

Any potential $\phi_j = N_j \tilde{\phi}_j$ 62 allowed by the quantum condition must also satisfy Eqn.s. (3.15). Even so, Eqn.s. (3.15), (4.17) and (4.18) represent three stages of differentiation, starting with the gauge function, f. In looking for the restrictions Eqn.s. (3.15) place upon the quantized potentials, we may employ a technique of mathematics in the solution of differential equations. We try to find a solution in the form of the product of functions of the separate variables. However, our trial solution must produce potentials of the form in Eqn. (4.16). Therefore, suppose we try to find a solution of the form

$$ln f^{\frac{1}{2}} = FG \tag{4.19}$$

where

$$F = f_t f_r f_\theta f_\phi f_\gamma \quad ,$$

with f_t = function of time only, f_r = function of spherical radius only, etc., and the function G is defined by the system of partial differential equations,

$$\frac{\partial G}{\partial x^j} = \frac{(N_j - G)}{F} \frac{\partial F}{\partial x^j} \quad (no\ sum) \quad .$$

The definition of the gauge potential, using the trial solution of Eqn. (4.19), now produces

$$\phi_j = \frac{\partial\, ln f^{\frac{1}{2}}}{\partial x^j} = \frac{\partial (FG)}{\partial x^j} = G \frac{\partial F}{\partial x^j} + F \frac{\partial G}{\partial x^j} \quad ,$$

but by the defining relations for G this becomes

$$\phi_j = G \frac{\partial F}{\partial x^j} + (N_j - G) \frac{\partial F}{\partial x^j} = N_j \frac{\partial F}{\partial x^j} \quad (no\ sum).$$

If we define

$$\tilde{\phi}_j = \frac{\partial F}{\partial x^j} \quad ,$$

then we have the proper form

$$\phi_j = N_j \tilde{\phi}_j \quad (no\ sum). \tag{4.20}$$

We may now use our trial solution to write the potentials in spherical coordinates:

$$\phi_1 = \frac{\partial ln f^{\frac{1}{2}}}{\partial r} = N_1 f_t f_{r'} f_\theta f_\phi f_\gamma = N_1 F_1 ,$$

$$\phi_2 = \frac{1}{r}\frac{\partial ln f^{\frac{1}{2}}}{\partial \theta} = \left(\frac{N_2}{r}\right) f_t f_r f_{\theta'} f_\phi f_\gamma = \left(\frac{N_2}{r}\right) F_2 ,$$

$$\phi_3 = \left(\frac{1}{rsin\theta}\right)\frac{\partial\, ln f^{\frac{1}{2}}}{\partial \phi} = \left(\frac{N_3}{rsin\theta}\right) f_t f_r f_\theta f_{\phi'} f_\gamma = \left(\frac{N_3}{rsin\theta}\right) F_3 ,$$

$$\phi_4 = a_0 \frac{\partial ln f^{\frac{1}{2}}}{\partial \phi} = a_0 N_4 f_t f_r f_\theta f_\phi f_{\gamma'} = a_0 N_4 F_4 ,$$

$$\phi_0 = \frac{\partial\, ln f^{\frac{1}{2}}}{\partial (ilc)} = \left(\frac{N_0}{ic}\right) f_t f_r f_\theta f_\phi f_\gamma = \left(\frac{N_0}{ic}\right) F_0 \quad , \tag{4.21}$$

where the notation $f_{t'}$ 72 denotes df_t/dt and F_j denoted

$$F_0 = \frac{dF}{dt} \quad ,$$
$$F_1 = \frac{dF}{dr} \quad ,$$
$$F_2 = \frac{dF}{d\theta} \quad ,$$
$$F_3 = \frac{dF}{d\phi} \quad ,$$

and

$$F_4 = \frac{dF}{d\gamma} \quad .$$

Substituting the potentials given by Eqn. (4.21) into the definition of the field tensor and using the required covariant differentiation, we obtain the field components

$$E_r = \frac{(N_1 - N_0)}{c} F_{01} \quad ,$$
$$E_\theta = \frac{(N_2 - N_0)}{cr} F_{02} \quad ,$$
$$E_\phi = \frac{(N_3 - N_0)}{crsin\theta} F_{03} \quad ,$$
$$V_4 = \frac{a_0 (N_4 - N_0) F_{04}}{c} \quad ,$$
$$B_r = \frac{(N_2 - N_3)}{rsin\theta}\left[\frac{F_{23}}{r} - N_3 cot\theta\, F_3\right] \quad , \qquad (4.22)$$
$$B_\theta = \frac{(N_3 - N_1)}{rsin\theta}\left[F_{13} - \frac{N_3 F_3}{r}\right] \quad ,$$
$$B_\phi = \frac{(N_1 - N_2)}{r}\left[F_{12} - \frac{N_2 F_2}{r}\right] \quad ,$$
$$V_r = (N_1 - N_4)\, a_0 F_{14} \quad ,$$
$$V_\theta = (N_2 - N_4)\, a_0 F_{24} \quad ,$$

and

$$V_\phi = (N_3 - N_4)\, a_0 F_{34}$$

These field components reflect the quantization of the potentials. However, the quantization of the fields is not a simple quantization because each component depends upon the difference of quantum numbers.

The field components given by Eqn. (4.22) must satisfy the differential equations of Eqn. (3.15). Therefore, if we substitute Eqn. (4.22) into Eqn. (3.15), we will obtain the restrictions upon the quantum numbers, N_j, and the functions f_t, f_r, f_θ, f_φ, and f_γ required for these fields to be the fields of a fundamental particle. We begin with the equation

$$\overline{\Delta} \bullet \overline{B} = 0 \quad . \qquad (3.15a)$$

This equation becomes

$$\left(\frac{1}{r_2}\right)\frac{\partial B_r r_2}{\partial r} + \left(\frac{1}{rsin\theta}\right)\frac{\partial(sin\theta\, B_\theta)}{\partial\theta} + \left(\frac{1}{rsin\theta}\,\partial B_\phi = 0\right) \quad ,$$

in spherical coordinates. Substituting from Eqn. (4.22) and simplifying, we finally arrive at

$$\left(\frac{1}{r_3 \sin\Theta}\right)\left\{(N_3 - N_2) N_3 r \cot\theta (f_3 + r f_{31}) + F_{23}\left[(N_2 - N_1) N_2 + (N_1 - N_3) N_3\right]\right\} = 0$$

This requires that

$$(N_2 - N_3) N_3 r \cot\theta (F_3 + r F_{31} = F_{23}[(N_2 - N_1) + (N_1 - N_3) N_3] \quad .$$

Substituting from the definition of the F_j, we find

$$(N_2 - N_3) N_3 \cot\theta \left[r f_t f_r f_\theta F_\phi f_\gamma + r^2 f_t f_{r'} f_\theta f_{\phi'} f_\gamma \right] = f_t f_r f_{\theta'} f_{\phi'} f_\gamma \left[(N_2 - N_1) N_2 + (N_1 - N_3) N_3 \right] \; .$$

If $f_t f_{\phi'} f_\gamma \neq 0$ 78, this may be rewritten as

$$(N_2 - N_3) N_3 \cot\theta \left(r f_r + r_2 f_{r'} \right) f_\theta = f_r f_{\theta'} \left[(N_2 - N_1) N_2 + (N_1 - N_3) N_3 \right]$$

However, we may divide by $f_r f$ cot and separate the equation into

$$\frac{[r f_r + r^2 f_{r'}]}{k f_r} = \frac{f_{\theta'}}{\cot\theta \, f_\theta} \; , \tag{4.23}$$

where

$$K = \frac{(N_2 - N_1) N_2 + (N_1 - N_3) N_3}{(N_2 - N_3) N_3} \; .$$

The left-hand side of Eqn. (4.23) is a function of r only, while the right-hand side is a function of θ only. Therefore, Eqn. (4.23) must be a constant. We can then write

$$\frac{[r f_r + r^2 f_{r'}]}{f_r} = \frac{f_{\theta'} k}{\cot\theta \, f_\theta} = \lambda_0 k = \lambda_n \; , \tag{4.24}$$

where the constant λ_n depends upon the set of quantum numbers, N_1, N_2, and N_3, for the particle. Thus, λ_n depends on the particle under consideration.

The radial equation in Eqn. (4.24) may be integrated immediately with the result

$$f_R = \left(\frac{k}{r} \right) e^{-\left(\frac{\lambda_n}{r} \right)} \; . \tag{4.25}$$

The appearance of this exponential functional form for the radial dependence is surprising and, at first, pleasing. The surprise is that this functional form comes only from the gauge function, f, playing the guiding role in the gauge fields and the field equation

$$\overline{\Delta} \bullet \overline{B} = 0 \; ,$$

which is a purely classical equation. Thus, the exponential neo-coulombic radial function does not appear at first to depend upon the fifth dimensionality, but only upon the quantum condition so that even a four-dimensional approach would have produced this same radial function.

It is pleasing to see the appearance of the exponential neo-coulombic radial functional of Eqn. (4.25) because the electron catastrophe has haunted theoreticians since the inverse radial dependence of the columbic potential was first seen. The radial function in Eqn. (4.25) is well behaved everywhere; as $r \to \infty$, $f_r \to 0$ and as $r \to 0$, $f_r \to 0$. A quick glance at the function might cause one to think it is the Yukawa potential, but a closer look will show that the exponent is the inverse of the exponent in the Yukawa potential.

The value for f_θ may be obtained by integrating the remaining portion of Eqn. (4.24). This integration produces

$$f_\theta = k_3 (\sin\theta)^{\lambda_0} \quad .$$

If the exponential neo-coulombic function corresponds to reality, then λ_n must be small, less than 10^{-17}m for electrons, and of the order of magnitude of 10^{-15}m for protons. Therefore, λ_0 must be very small, which in turn implies f_θ is very close to a constant.

The equations resulting from substituting the quantized potentials into the remaining non-source equations [3.15b], [3.15e], and [3.15f], produce the following restrictions

$$[3.15b] \to \begin{cases} (N_2 - N_3) N_3 f_{\phi'} f_{\gamma'} = 0 \\ (N_3 - N_1) N_3 f_{\phi'} f_{\gamma'} = 0 \\ (N_1 - N_2) N_2 f_{\theta'} f_{\gamma'} = 0 \end{cases}$$

$$[3.15e] \to \begin{cases} (N_2 - N_3) N_3 f_{t'} f_{\phi'} = 0 \\ (N_3 - N_1) N_3 f_{t'} f_{\phi'} = 0 \\ (N_1 - N_2) N_2 f_{t'} f_{\theta'} = 0 \end{cases}$$

[3.15f] (satisfied identically).

When the potentials are substituted into the equations with source terms, Eqn.s [3.15c-e] and [3.15h], the resulting equations are very complex. To reduce the complexity of the equations, the assumption was made that all source terms were zero; that is,

$$\rho = 0; \bar{J} = \bar{0}; J_4 = 0 \quad .$$

This assumption reduced the complexity somewhat but still left a system of equations that, thus far, is unsolved. However, an interesting aspect of this assumption is the possible existence of a radial electric field without the presence of any electric charge within, or upon, the particle. This possibility rests upon the pressure of the term $\partial V_4 / \partial \gamma$ 86 in the Eqn. [3.15d].

Much was learned about the interaction of charged particles by considering only the radial dependence of the electric field while temporarily neglecting the magnetic field or any potential variation of the electric field with azimuthal angles. This latter is the spherically symmetric field assumption. Having not yet obtained a complete solution to the system of equations that is the result of substituting the quantized fields into the eight field equations, it proved beneficial to make the assumption of spherically symmetric fields in which the only variation of the fields is the radial dependence specified by the neo-coulombic radial function.

Then, if we want to explore the radial dependence of static forces between the fundamental particles allowed by the quantum condition, we must consider the force law,

$$K_j = \left(\frac{1}{c}\right) F_{jk} J_k \quad ,$$

whose spatial components may be written

$$\overline{K} = \rho\overline{E} + \frac{1}{c}(\overline{J} \; x \; \overline{B}) + \left(\frac{J_4}{c}\right)\overline{V} \quad .$$

By restricting our concern to static forces we can concentrate on the force density,

$$\overline{K} = \rho\overline{E} + \frac{J_4}{c}\overline{V} \quad . \qquad (4.26)$$

Thus, the radial dependence of the electric field, E, and the V field are all that need to be considered at the moment. Substituting the radial function, Eqn. (4.25), into the field expressions, Eqn. (4.22), we find

$$E_r = \left(\frac{Zk}{r^2}\right)\left(1 - \frac{\lambda_n}{r}\right)e^{-\left(\frac{\lambda_n}{r}\right)} \cdot \qquad (4.27)$$

and

$$V_r = \left(\frac{Wg}{r^2}\right)\left(1 - \frac{\lambda_n}{r}\right)e^{-\left(\frac{\lambda_n}{r}\right)} \cdot$$

where Z = (N_1-N_0) and W = (N_1-N_4) so that the quantum number Z appears in the radial electric field the same as it does classically.

From Eqn. (4.27), the electric field of fundamental particles allowed by the Dynamic Theory is quantized by the quantum condition, and the quantum steps may only be integer steps. This would necessarily preclude any particles with fractional charge steps.

Substituting the radial fields of Eqn. (4.27) into the force law, Eqn. (4.26), and integrating the charge density over the physical extent of the particle, we find the radial force between two particles is

$$F = \left(\frac{q_1 Zk}{r^2}\right)\left(1 - \frac{\lambda_n}{r}\right)e^{-\left(\frac{\lambda_n}{r}\right)} + \left(\frac{g_2 Wg}{r_2}\right)\left(1 - \frac{\lambda_n}{r}\right)e^{-\left(\frac{\lambda_n}{r}\right)} \cdot \qquad (4.28)$$

where

$$q_1 = \int \rho d(Vol) \quad , \quad g_2 = \int\left(\frac{J_4}{c}\right)d(Vol) \quad .$$

If we consider the electric force in Eqn. (4.28) and restrict our attention to r such that r >> λ_n, then the electric force becomes the columbic force

$$F_E = \frac{q_1 Zk}{r^2} \quad .$$

Further, the other force term, based upon the V field, may be seen to also have the same long-range form,

$$F_V = \frac{g_2 Wg}{r^2} \quad .$$

Thus, this force is also an inverse square long range force. Therefore, the nonelectric force in Eqn. (4.28) cannot be a nuclear force. What then must be the interpretation of this force, or must its appearance be interpreted to mean that nature cannot have a five-dimensional character?

The only force known in nature that has a long-range character, in addition to the electrostatic force, is the gravitational force. But how can we interpret the V field as the gravitational field when Einstein showed that the gravitational field could be explained by a vector curvature in a four-dimensional manifold and the V field is a gauge field in a five-dimensional manifold? If the V field were to be considered as gravitational, then the bending of light around the sun, predicted by Einstein's General Theory of Relativity, must have another explanation. Is this possible? If the V field is gravitational, then is there room in the Dynamic Theory for an explanation of nuclear phenomena or must it also follow the current approach to nuclear physics, thereby requiring similar additional assumptions?

These and other questions occur when we see the long-range character of the V field force. It is this theoretical quandary, presented by the V field, that spoils the pleasant surprise of seeing a non-singular electrostatic field emerge from the quantum condition. A number of possibilities appear dependent upon the answers obtained to the previous questions and/or others. Primarily, the possibilities pertain to the validity of the five-dimensional view, for it is from this five-dimensionality that the V field comes. One reasonable approach, in attempting to find a possible way out of this quandary, seems to be to suppose the gravitational interpretation is a possibility and then see how the Dynamic Theory compares with measurable experimental evidence.

4.10 Self-Energy of Charged Particles.

One of the difficulties in Maxwellian electromagnetism is the infinite self-energy that is predicted for a charged particle. This "electron catastrophe," or singularity, does not exist with the non-singular neo-coulombic field, and the self-energy of a charged particle may be found.

In classical electromagnetic theory the self-energy of a charged particle is discussed but its value has not been established. This is because the expression for the self-energy is a function of the radius associated with the physical extent of the charge distribution. Thus, the radius of the charged particle must be known before the self-energy can be determined.

Currently the self-energy of a charged particle is equated with the energy associated with its inertial mass by $E = mc^2$. Then the radius associated with its energy is taken as the "radius" of the particle. There is no intention that this radius be the physical radius of the particle though it compares favorably with experimental values.

The question arises here of whether or not the Dynamic Theory, with the five-dimensional viewpoint, can theoretically predict the self-energy and/or the radius of the physical extent of the mass or charge distribution of the particle. One of the beneficial aspects of the generalization of physical theory as done in the Dynamic Theory is the possibility of using conceptualizations and procedures developed in one branch of physics in another branch. This aspect of the theory appears applicable here. The self-energy of a charged particle is the notion that a certain amount of energy be associated with the existence of the particle and its charge. This notion may be associated with the notion of free energy used in thermodynamics, for, if the self-energy of the charged particle is its free energy, then it represents the energy which may be "freed" upon converting the particle into energy. Conversely, this would represent the energy required to assemble the charged particle.

With the conceptualization of free energy the second law provides the condition for a stable equilibrium state, namely that a charged particle in an equilibrium state must exist at a minimum of its free energy. Thus, if the self-energy, or free energy, of a charged particle is sought, then minimizing its free energy will yield the desired result.

The free energy was defined, in analogy with the thermodynamic case, as

$$G \equiv U - \phi S - x^{\alpha} F_{\alpha}, \tag{4.29}$$

where α depends upon the applicable work terms which here will be taken as the three spatial dimensions, so that $\alpha = 1, 2, 3$. The first law is given by

$$\bar{d}E = dU - F_{\alpha} dx^{\alpha}$$

while the second law yields

$$\phi dS = dU - F_{\alpha} dx^{\alpha}$$

for a quasi-static, reversible process. Therefore, the differential change in the system energy is

$$dU = \phi dS + F_{\alpha} dx^{\alpha}. \tag{4.30}$$

Differentiating Eqn (4.29) gives the differential change in the free energy as

$$dG = dU - \phi dS - Sd\phi - F_{\alpha} dx^{\alpha} - x^{\alpha} dF_{\alpha}. \tag{4.31}$$

Substituting Eqn (4.30) into (4.31) yields

$$dG = - Sd\phi - x^{\alpha} dF_{\alpha}. \tag{4.32}$$

The force in Eqn (4.32) is considered to be the Lorentz force

$$F_{\alpha} = q[\overline{E} + (\overline{v}x\overline{B})]_{\alpha}$$

so that Eqn (3.48) becomes

$$dG = - Sd\phi - x^{\alpha} d\{ q[\overline{E} + (\overline{v}x\overline{B})]_{\alpha}\}.$$

If we wish to consider the change in free energy with respect to a change in the charge at a constant velocity, we find that, since ρ is a function of velocity only, dρ= 0. The specification of constant velocity stems from the desire to obtain the self-energy of a charged particle; therefore, the particle should be considered as sitting still, so that it will have no kinetic energy. The differential change of free energy for a stationary particle is then

$$dG = - Sd\phi - x^{\alpha} d\{q[\overline{E} + (\overline{v}x\overline{B})]_{\alpha}\},$$

$$= - Sd\phi - x^{\alpha} \{dq[\overline{E} + (\overline{v}x\overline{B})]_{\alpha} + qd[\overline{E} + [(\overline{v}x\overline{B})]_{\alpha}\}.$$

so that for ρ = constant

$$\left(\frac{\partial G}{\partial q}\right)_{\phi} = - x^{\alpha} [\overline{E} + (\overline{v}x\overline{B})]_{\alpha} - x^{\alpha} q\left(\frac{\partial[\overline{E} + (\overline{v}x\overline{B})]_{\alpha}}{\partial q}\right)_{\phi}.$$

but

$$\overline{E} + (\overline{v}x\overline{B})$$

is independent of the charge q and, therefore,

$$\left(\frac{\partial G}{\partial q}\right)_{\phi} = - x^{\alpha} [\overline{E} + (\overline{v}x\overline{B})]_{\alpha}.$$

If the charge is not in motion, then

$$\left(\frac{\partial G}{\partial q}\right)_{\phi} = - x^{\alpha} E_{\alpha} \tag{4.33}$$

since v = 0.

If G is the self-energy of a charged particle, then by Eqn. (4.32)

$$\frac{\partial G}{\partial q} = - r\, E_r \quad ,$$

the change in the self-energy is given with respect to a change in the charge q. If we assume a spherically symmetric charge density, ρ, then

$$dq = \rho dv = 4\pi\, r^2\, \rho dr \quad .$$

We may then find the free energy by the integration

$$\int_{G_0}^{G} dG = - \int_{q_0}^{q} r\, E_r\, dq$$

$$= - \int_{0}^{R} 4\pi\, r^3\, E_r\, \rho dr \quad ,$$

where R represents the radius within which the charge density ρ is contained.

The field Eqn. [3.15d] is

$$\overline{\Delta} \bullet (\varepsilon \overline{E}) = 4\pi\rho - a_0 \frac{\partial(\varepsilon V_4)}{\partial\gamma} \quad .$$

The entire right-hand side of this field equation behaves as a charge density; therefore, we may equally write

$$\overline{\Delta} \bullet (\varepsilon \overline{E}) = 4\pi\rho \quad , \tag{4.34}$$

where it is understood that either $\partial(\varepsilon V_4)/\partial\gamma$ 109 is zero or ρ is considered to be a total effective charge density. In either event, Eqn. (4.34) gives us

$$\left(\frac{\varepsilon}{r^2}\right) \frac{\partial(r^2 E_r)}{\partial r} = 4\pi\rho \quad ,$$

when we consider only a radially symmetric field E_r. Thus,

$$\rho dr = \left(\frac{\varepsilon}{r\pi\, r^2}\right) d(r^2 E_r) \quad . \tag{4.35}$$

Substituting Eqn. (4.35) into Eqn. (4.33) the self-energy is then found to be

$$G = - \varepsilon \int (r\, E_r) d(r^2 E_r) + G_0 \quad . \tag{4.36}$$

If we now use the neo-coulombic electric field given by

$$E_r = \frac{e}{4\pi\varepsilon r^2}\left(1 - \frac{\lambda}{r}\right)e^{-\left(\frac{\lambda}{r}\right)}$$

in the integral of Eqn. (4.36), we find

$$G = -e\int_0^R \frac{e}{4\pi\varepsilon r}\left(1 - \frac{\lambda}{r}\right)e^{-\left(\frac{\lambda}{r}\right)} d\left[\left(\frac{e}{4\pi\varepsilon}\right)\left(1 - \frac{\lambda}{r}\right)e^{-\left(\frac{\lambda}{r}\right)}\right] + G_0 \quad ,$$

which may be integrated so that

$$G = \left[\frac{-e^2}{2(4\pi)^2\varepsilon\lambda}\right]\left[\left(\frac{\lambda}{R}\right)^3 + \frac{3}{4}\left(\frac{\lambda}{R}\right)^2 - \frac{1}{2}\left(\frac{\lambda}{R}\right) - \frac{1}{4}\right]e^{-\left(\frac{2\lambda}{R}\right)} + G_0 \quad . \tag{4.37}$$

To find the specific value of the self-energy, we must find the R that minimizes G. Therefore, set

$$\frac{\partial G}{\partial R} = 0 \quad .$$

After carrying out the required differentiation and simplifying, this is satisfied if

$$R^2 + \left(\frac{3}{5}\lambda\right)R - \left(\frac{2}{5}\lambda\right) = 0 \quad . \tag{4.38}$$

Equation (4.38) only has one positive root, which is

$$R = 0.4\lambda \quad .$$

Substituting this result into Eqn. (4.37), the self-energy becomes

$$G = \left[\frac{-e^2}{(4\pi)^2\varepsilon\lambda}\right](0.063379) + G_0 \quad ,$$

or

$$G = \frac{-7.26235 \times 10^{-3}\ MeV - fermi}{\lambda\ (fermi)} + G_0 \quad , \tag{4.39}$$

when λ is given in units of fermi.

An example may be a proton for which λ is approximately 1 fermi if the proton-proton scattering data is considered. Then if λ~fermi,

$$G_p = -7.26235 \times 10^{-3}\ \text{MeV} + G_{op} \ ,$$

so that

$$G_{op} = 938.263\ \text{MeV} \tag{4.40}$$

is the part of the proton rest energy independent of its charge. The charge energy of the proton would then be

$$G_{cp} = -7.26235 \text{ keV} ,$$

which is negligibly small compared to the non-charge energy G_{op}.

What is the nature of the energy G_{op}? It is not energy caused by the presence of electric charge on the protons. Also the self-energy, G, was found for a resting particle. If we associate the resting self-energy, G, with the rest mass as

$$G_p = m_{op}c^2 = G_{cp} + G_{op} ,$$

and G_{cp} is the portion of the proton's rest energy that is due to its charge, then G_{op} must be that portion of the rest energy that is due to the proton mass above. In this case, the proton mass energy, G_{op}, is given by Eqn. (4.40).

Suppose we consider an electron and assume that λ_e - ~ 10^{-3} fermi. Then

$$G_{e-} = 0.511 \text{ MeV} = -7.26235 \text{ MeV} + G_{0e-},$$

or the mass energy of the electron would then be

$$G_{oe-} = 7.773 \text{ MeV} ,$$

whereas its charge energy is

$$G_{ce-} = -7.26235 \text{ MeV}$$

4.11 Nuclear Phenomena.

The electrostatic force, appearing in Eqn. (4.27), differs significantly from the columbic force only when r becomes small enough to be of the order of magnitude of the λ_n. The first experimental evidence that the scattering of charged particles by other charged particles was not always columbic was the Rutherford scattering data. The appearance of the exponential multiplier in the neo-coulombic force of Eqn. (4.27) prompts us to ask whether or not the difference between this force and the columbic force suffices to explain nuclear phenomena without resorting to the postulation of a new short-range force such as the nuclear force.

An obvious starting point to explore the possibility that the neo-coulombic force might apply to nuclear phenomena would probably be the

Rutherford scattering formula. This may be done; however, the appearance of the exponential term makes an analytical expression difficult, if not impossible, to obtain. We may arrive at a solution of limited usefulness if we assume that r >> λ. Further, in considering particle scattering, we shall restrict our consideration to scattering of like particles, only, so we are guaranteed that only one λ is involved.

The best way to investigate the scattering cross sections is to start with the solutions of the equations of motion for planetary orbits in which the force is given by the neo-coulombic force instead of the simple inverse r^2 force from Newton's gravitational force. We find the radial equation becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{Mk}{L^2}(1-\lambda u)e^{-\lambda u},$$

where u = 1/r, k is the gravitational constant, and L is the orbital angular momentum. The exponential function may be expressed in terms of a power series, and our radial equation becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{Mk}{L^2}(1-\lambda u)\left[1+(-\lambda u)+\frac{(-\lambda u)^2}{2!}+\ldots+\frac{(-\lambda u)^n}{n!}+\ldots\right].$$
$$=\frac{Mk}{L^2}\left[1+\sum_{(n=1)}^{\infty}\frac{(n+1)}{n!}(-\lambda u)^n\right]$$

by assuming r>>λ so that λ/r = λu <<1, then we may neglect the terms with n≥3 in our radial equation. The result of this assumption is

$$\frac{d^2u}{d\theta^2}+\alpha^2u=\frac{Mk}{L^2}+\left(\frac{3\lambda^2Mk}{2L^2}\right)u^2$$

with

$$\alpha^2=1+\frac{2\lambda Mk}{L^2}.$$

This equation may be compared with the classical equation,

$$\frac{d^2u}{d\theta^2}+u=\frac{Mk}{L^2},$$

or with the general relativistic equation[6],

$$\frac{d^2u}{d\theta^2}+u=\frac{Mk}{L^2}+3Mku^2,$$

which has the identical form of our equation. Thus, the same method of perturbations may be used to obtain a solution as was used for the relativistic case. The result of this calculation is the solution,

$$u = \frac{1}{r} = \frac{Mk}{L^2}\left[1 + e\cos(\alpha\theta - \theta_0 - \delta\theta_0)\right], \tag{4.41}$$

where

$$\delta\theta_0 = \frac{3\lambda Mk^2\theta}{2L^2}$$

is the increase in the perihelion.

Notice we have shown that the neo-coulombic force will predict an advance in the perihelion of planetary orbits with the solutions to our planetary orbits equation. We will discuss this further at a later time. We needed the solution given by Eqn (4.41) in order to obtain an expression for the scattering cross section of like particles.

If we now consider the solution, Eqn. (4.41), obtained with the assumption that r >> λ, then the scattering cross section may be expressed as

$$d\sigma = \left(\frac{q_1 q_2}{2mv_0^2}\right)\left[\frac{2\pi\sin\theta d\theta}{\sin^4\left(\frac{\theta}{2}\right)}\right]\delta \quad , \tag{4.42}$$

where

$$\delta \equiv \left\{\frac{1 + 6\left(\frac{\lambda E}{k}\right)^2 \sin^4\left(\frac{\theta}{2}\right)\left[1 + 2^{(\pi-\theta)}\tan\left(\frac{\theta}{2}\right)\right]}{\left[1 + \left(\frac{3}{2}\right)\left(\frac{4\lambda E}{k}\right)^2 \sin^2\left(\frac{\theta}{2}\right)\sin\theta(\pi - \theta)\right]^4}\right\} .$$

The appearance of the factor δ expresses the first-order deviation of the scattering cross section of the neo-coulombic force from that of the columbic force. However, the assumption that r >> λ implies a limit on the minimum impact parameter for which this cross section retains validity. Therefore, a computer solution is probably necessary to really investigate the scattering of charged particles using the neo-coulombic force.

Figure 6. Comparison of coulomb and neo-coulomb forces at short range.

Another way of visualizing the neo-coulombic force is to make a plot of it and compare it with a plot of the columbic force. Figure 6 compares these two forces plotted with the separation variable in fermions and normalized so that the columbic force at one-fermion separation is unity. Note that this plot compares the forces for like particles to ensure that λ is the same for both particles. Figure 6 shows that the neo-coulombic force is virtually indistinguishable from the coulomb force for separations greater than approximately 10λ. However, at a separation of exactly λ, the force is identically zero. In terms of the classical notion of nuclear forces, we would say that at separations greater than 10λ, the nuclear force is negligible, whereas at a separation of λ the magnitude of the nuclear force was equal to the magnitude of the coulomb force. The neo-coulombic force becomes an attractive force for separations less than λ. This is exactly the behavior to be expected of a non-singular potential. For a potential to be non-singular it must tend to zero as r goes to zero. Such a potential which tends to zero for r tending to zero and for r tending to ∞ must have a maximum absolute value in between. At that maximum the force, being determined by the slope of the potential, will go to zero and will be of the opposite signs on each side of the zero.

Now let us look at the force between unlike particles, say a proton and an electron. Consider the electron and proton to be placed on a horizontal surface separated by a distance, r, with the proton to the right of the electron. Thus, the long-range attractive forces between these two particles will cause the proton to experience a force to the left while the electron will experience a force to the right. We may than write the force on the proton that is due to the positive charge of the proton being in the electron field as

$$\begin{aligned}\overline{F}_p &= q_p \overline{E}_e \\ &= \frac{-k}{r^2}\left(1 - \frac{\lambda_e}{r}\right) e^{-\left(\frac{\lambda_e}{r}\right)} \ (\hat{u}_x) \ ,\end{aligned} \tag{4.43}$$

where the electron field involving the electron lambda has been accounted for. The electron force owing to the electron charge being in the proton field is given by

$$\begin{aligned}\overline{F}_e &= q_e \overline{E}_p \\ &= \frac{k}{r^2}\left(1 - \frac{\lambda_p}{r}\right) e^{-\left(\frac{\lambda p}{r}\right)} (\hat{u}_x) \ .\end{aligned} \tag{4.44}$$

Figure 7 plots both these forces as a function of the separation, r, where, $\lambda_p = 10^{-15}$m, or λ_p = 1 fermi has been assumed. The electron-electron scattering data show that the electron-electron interaction behaves in a coulombic manner even when separations are approximately 0.01-0.1 fm. To be consistent with this data, we have assumed $\lambda_e = 10^{-3}$ fermi.

From this plot of the force on the proton and the force on the electron, we see that for separations less than about 10 fermis the forces become extremely unsymmetrical. This immediately and visually demonstrates that the neo-coulombic exponential force violates Newton's third law requiring that the force on the proton be equal in magnitude and opposite indirection to the force on the electron. The question arises whether or not a violation of Newton's third law has ever been seen as the result of an interaction between an electron and a proton? The answer, based on a neutron disintegration from which a proton and electron emerge, is definitely yes; Newton's third law was seen to be violated. To reinstate Newton's third law in neutron disintegration and all other beta decay, Pauli postulated the existence of the neutrino. Fermi later developed his theory of weak interactions,[11] from which appeared the necessity to talk of a fourth force in nature.

Can it be that the neo-coulombic force, which requires distinct λ for distinct fundamental particles, accounts for the action of the weak forces also? The possibility that it might opens the theoretical flood gates and a virtual tidal wave of questions surges forth. Does this mean the neutrino does not exist? What about the experimental evidence submitted in support of the capture of a free neutrino?[12] Could this mean that the neutron might be bound states of an electron and proton? This question should be followed by, what about conservation of angular momentum in neutron decay (i.e., spin), conservation of linear momentum, and Heisenberg's uncertainty principle?

Figure 7. Neo-coulombic forces between unlike particles at short range.

The preceding questions do not begin to scratch the surface of the theoretical questions that need to be answered as the result of considering the possibility that the force law of Eqn. (4.27) with only a gravitational force plus the neo-coulombic force might explain the phenomena now thought to require four distinct forces in their explanation. However, the appearance of a non-singular force with the apparent range of the neo-coulombic form cannot be thrown out offhand. Therefore, it seems that the only reasonable choice is to systematically and thoroughly explore the possibilities.

If we again consider the plots of the proton and electron forces in Fig. 7. we see that, at atomic separations and greater distances, the forces obey Newton's third law and the difference between the neo-coulombic and columbic forces is so small that it could not be detected in atomic or macroscopic phenomena. But as the separation becomes smaller, the picture begins to change. When the r approaches λ_p, the electron is no longer attracted to the proton as strongly as the proton is attracted to the electron. If the separation is exactly λ_p, then the electron is indifferent to the proton's presence. The proton, on the other hand, is still very much attracted to the electron. If for the moment, we ignore the interpretation of Heisenberg's uncertainty principle that would say it cannot be, then we could easily imagine a circular proton orbit around a stationary electron, during which the proton stays at a radius of λ_p from the electron. The electron should be stationary during such motion because it would experience no force.

We now consider a separation between the electron and proton, which is some simple fraction of λ_p. Here, we find the electron repulsed by the proton, but the proton is still attracted to the electron. Notice that the force on both particles, from our initial positioning of the proton on the right, is to the left. If both particles were given an angular momentum such that they were placed into synchronized circular orbits, then because their synchronous motion always results in the force on both particles being directed along the line separating them and from the proton toward the electron or from the electron away from the proton then, again ignoring arguments from the uncertainty principle, circular orbits in which the electron is in a small orbit about a space point could be imagined, where the proton is in a much larger orbit about the same space point.

Let us follow this picture a little farther and write simple Newtonian-like force laws for this situation. The situation envisioned is presented in Fig. 3. The electron position is given by r_e from the origin, and the position of the proton is given by r_p. The separation between them is

$$r = r_p - r_e \ . \tag{4.45}$$

Because the force is always directed along the line separating the two particles, we may write the radial equation of motion for the proton as

$$m_p \frac{v_p}{r_p} = \frac{-k}{r^2}\left(1 - \frac{\lambda_e}{r}\right) e^{-\left(\frac{\lambda e}{r}\right)} \ , \tag{4.46}$$

where the assumed circular motion has been taken into account and v_p is the tangential proton velocity. The electron equation of motion is given by

$$m_e \frac{v_e}{r_e} = \frac{k}{r^2}\left(1 - \frac{\lambda_p}{r}\right) e^{-\left(\frac{\lambda_p}{r}\right)}. \tag{4.47}$$

Figure 8. Electron and proton orbits.

In both equations $k < 0$.

The right-hand side of Eqn. (4.46) and (4.47) are both functions of the separation, r, whereas the two left-hand sides are individually functions of r_e and r_p. A solution is possible only when the three equations, Eqn.s (4.45)-(4.47), are solved simultaneously.

An alternative approach is to add Eqn. (4.46) and (4.47) to obtain the equation of motion for the center of mass,

$$m_p \frac{v_p 2}{r_p} + m_e \frac{v_e 2}{r_e} = \frac{k}{r^2}\left(1 - \frac{\lambda_p}{r}\right) e^{-\left(\frac{\lambda_p}{r}\right)} - \left(1 - \frac{\lambda_e}{r}\right) e^{-\left(\frac{\lambda_p}{r}\right)} \ , \tag{4.48}$$

or

$$\frac{MV^2}{R} = \frac{k}{r^2}\left(1-\frac{\lambda_p}{r}\right)e^{-\left(\frac{\lambda^p}{r}\right)} - \left(1-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_p}{r}\right)} \ ,$$

where R = $(m_p r_p + m_e r_e)/(m_p + m_e)$ and M = $m_p + m_e$. From Eqn. (4.48) we see that bound states, where the center of mass is in motion as the result of the asymmetrical force, may only occur when the separation is less than λ_p.

All of these equations of motion exhibit a feature not usually found in equations of motion. That is, because the force depends on the separation, r, between the particles and not strictly on the position, r_e, for the election, then the usual integration of the force over a change of position, which produces the potential energy, cannot be readily done because,

$$\begin{aligned} V(r_e) &= -\int F_e \, dr_e \\ &= -\int \left(\frac{K}{r^2}\right)\left(1-\frac{\lambda_p}{r}\right)e^{-\left(\frac{\lambda_p}{r}\right)} dr_e \ . \end{aligned} \tag{4.49}$$

However, Eqn. (4.45) and Eqn. (4.47) may be used to obtain r_e as a function of r, or vice versa, so the integration of Eqn. (4.49) may be completed. The transcendental function in the force law prohibits an analytical solution of r_e as a function of r. Therefore, only numerical or graphic solutions of these equations are possible.

4.12 Heisenberg's Uncertainty Principle and Geometry.

The suggestion that bound states of electrons and protons might exist where the orbits are of the approximate order of magnitude of nuclear dimensions, is essentially a return to the notion that a neutron might be such a state. This idea gave way under arguments of conservation of momentum and Heisenberg's Uncertainty Principle to the view that electrons are forbidden to be found within the nucleus. Therefore, let us take another look at those fundamental tenets of quantum mechanics, the Poisson brackets.

The classical Poisson bracket is defined by

$$\{F,G\} = \sum_j \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right) \ ,$$

where F and G are any two functions of the canonically conjugate variables q_j and p_j. The special relations that occur when F and G are q_j and p_j, respectively, are especially important in quantum mechanics; these are, classically:

$$\{q_j, q_k\} = 0$$
$$\{p_j, p_k\} = 0 \qquad (4.50)$$
$$\{q_j, p_k\} = 0 ,$$

where δ_{jk} is the Kronecker delta. The classical Poisson brackets of Eqn. (4.50) are obtained when Euclidean spaces are assumed. However, the definition of Poisson brackets remains valid for general metric spaces, when the notion of covariant differentiation is used. If we now consider the momenta, expressed in a general coordinate system, the covariant components,

$$p_j = mg_{ij}\dot{x}^k$$

and (4.51)

$$p_j = g^{jl} p_l = m\dot{x}^j$$

are the contravariant components. Covariant differentiation must be carried out with respect to contravariant vector components. There, in a general space the canonically conjugate variables to be considered are x^j and p^k, and the Poisson bracket of the position and momenta becomes

$$\begin{aligned}\{x^j, p^k\} &= \left[\frac{\partial x^j}{\partial x^l} + \begin{Bmatrix} j \\ s\,l \end{Bmatrix} x^2\right]\frac{\partial p^k}{\partial p^l} - \frac{\partial x^j}{\partial p^l}\left[\frac{\partial p^k}{\partial x^l} + \begin{Bmatrix} k \\ n\,l \end{Bmatrix} p^n\right] \\ &= \left[\delta_{jl} + \begin{Bmatrix} j \\ s\,l \end{Bmatrix} x^s\right]\delta_{lk} \ .\end{aligned} \qquad (4.52)$$

or

$$\{x^j, p^k\} = \delta_{jk} + \begin{Bmatrix} j \\ s\,k \end{Bmatrix} x^s \ .$$

Quantum mechanics adopts the operator,

$$\left(\frac{\hbar}{i}\right)\frac{\partial}{\partial x^j} \rightarrow p_j \ ,$$

for the momentum. This, in general case, becomes the covariant operator

$$\left(\frac{\hbar}{i}\right)(\ \ )_{,j} \rightarrow p_j \ . \qquad (4.53)$$

The operator for the contravariant momentum components is then

$$\left(\frac{=}{i}\right) g^{jl}\,(\quad)_{,l} \rightarrow p^{j} \quad . \tag{4.54}$$

Now if we look at the quantum Poisson bracket, where the operators are operating on a scalar Ψ, then

$$\begin{aligned}
[\,x^{j}, p^{k}\,]\,\psi &= \left[x^{j}\left(\frac{=}{i}\right) g^{kl}\frac{\partial \phi}{\partial x^{l}} - \left(\frac{=}{i}\right) g^{kl}\,(\,x^{j}\psi\,),l \right] \\
&= x^{j}\left(\frac{=}{i}\right) g^{kl}\frac{\partial \psi}{\partial x^{l}} - g^{kl}\left(\frac{=}{i}\right)\left[\frac{\partial x^{j}}{\partial x^{l}} + \left\{ \begin{matrix} j \\ s\,l \end{matrix} \right\} x^{s}\right]\psi \\
&\qquad - \left(\frac{=}{i}\right) g^{jl}\, x^{j}\frac{\partial \psi}{\partial x^{l}} \\
&= i\hbar\, g^{kl}\left[\delta_{jl} + \left\{ \begin{matrix} j \\ s\,l \end{matrix} \right\} x^{s}\right]\psi \quad .
\end{aligned} \tag{4.55}$$

This may be written in terms of the classical Poisson bracket, Eqn. (4.52), as

$$\left[x^{j}, p^{k}\right]\psi = i\,\_\; g^{kl}\left\{x^{j}, p^{l}\right\}\psi \quad .$$

If the space is Euclidean, then the g^{kl} become the Kronecker delta and the Christoffel symbols vanish and the quantum Poisson bracket of Eqn. (4.55) becomes

$$\left[x^{j}, p_{k}\right]\psi = i\_\;\delta_{jk}\,\psi \quad ,$$

because $p^k = g^{kl}\, p_l = \delta^{kl} p_l = p_k$. However, from Eqns. (4.53) and (4.54), we see that the metric does play a role in the quantum operators. This should also be seen in the use of the operators in the Schrodinger Hamiltonian operator, because

$$\begin{aligned}
p_{j}\,p^{j} &= \frac{=}{i}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{j}}\left(\sqrt{g}\; p^{j}\right) \\
&= \frac{=}{i}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{j}}\left(\sqrt{G}\left(\frac{=}{i}\right) g^{jl}\frac{\partial}{\partial x^{l}}\right) \\
&= \frac{-^{2}}{\sqrt{g}}\frac{\partial}{\partial x^{j}}\left(\sqrt{g}\; g^{jl}\frac{\partial}{\partial x^{j}}\right)
\end{aligned} \tag{4.56}$$

becomes the operator to be used in a general space and, of course, is the operator used in applying Schrodinger's equation to the hydrogen atom.

The geometrical effect may be seen also in Dirac's equation by considering that the restrictions,

$$
\begin{gathered}
(\bar{\alpha} \bullet \bar{p})^2 = p_j p^j \quad , \\
\beta^2 = 1 \quad , \\
\text{and} \\
\bar{\alpha}\beta + \beta\bar{\alpha} = 0 \quad ,
\end{gathered}
\tag{4.57}
$$

must be met in order for solutions of

$$p^o |> = H |> \quad ,$$

where

$$H = -\left(\bar{\alpha} \bullet \bar{p} + \beta m\right) \quad ,$$

is also a solution of $p_j p^j = m^2$ in natural units.

The first restriction may be rewritten as

$$(\alpha\, p^j)^2 = p_i\, p^j = m^2\, g_{jk}\, \dot{x}^j\, \dot{x}^k = g^{jk}\, p^j\, p^k$$

by the definition of the momenta. Then, if we expand the left-hand side and equate coefficients of the $p^j p^k$, we find that

$$
\begin{gathered}
(\alpha)^2 = -g_{11} \ , \\
(\alpha)^2 = -g_{22} \ , \\
(\alpha)^2 = -g_{33} \ , \\
\alpha_1\alpha_2 + \alpha_2\alpha_3 = \{\alpha_1\alpha_2\} = -2\,g_{12} \quad , \\
\alpha_1\alpha_3 + \alpha_3\alpha_1 = \{\alpha_1\alpha_3\} = -2\,g_{13} \quad ,
\end{gathered}
\tag{3.77}
$$

and

$$\alpha_2\alpha_3 + \alpha_3\alpha_2 + \{\alpha_2\alpha_3\} = -2\,g_{23} \quad .$$

From Eqn. (3.77), for a Euclidian metric where $g_{jk} = \delta_{jk}$, these restrictions reduce to the usual restrictions. Any metric properties will affect these restrictions and will therefore feed into the solutions.

Now, of what benefit it this discussion of geometrical effect upon quantum mechanics in considering the neo-coulombic force? Recall that the neo-coulombic force came from a gauge function in a Weyl space. A gauge function has a geometrical effect that could be thought of as effectively changing the unit of action in quantum mechanics. To see the basis for this statement, let us recall the quantum Poisson bracket operations on a scalar,

$$\left[x^j, p^k\right]\psi = i\,\_\,g^{kl}\left[\delta_{jl} + \left\{ {j \atop s\,l} \right\} x^s\right]\psi \quad , \qquad (3.74)$$

and let us define

$$\_'\,\delta_{jk} = \_\,g^{kl}\left[\delta_{jl} + \left\{ {j \atop s\,l} \right\} x^s\right] \quad , \qquad (3.78)$$

then we can write

$$\left[x^j, p^k\right]\psi = i\,\_'\,\delta_{jk} \quad , \qquad (3.79)$$

which has the same form now used but the effective unit of action _′158 depends on the geometry as seen by Eqn. (3.78).

We may look at the effective unit of action in yet another way. Recall, from the principle of maximum entropy, that the generalized entropy is the action. Thus, quantization of the action is a quantization of the generalized entropy. But, because the entropy space is tied to the sigma space, we have

$$\left(dq^0\right)^2 = f(d\sigma)^2 \quad .$$

The gauge function is a function of the space point; therefore, the gauge function varies continuously from point to point in the space. Thus, if the generalized entropy is quantized, so must be σ. We may write

$$q^0 = n\_' = \sigma = n\_ \quad ,$$

where the difference between _′161 and _ 162 contains the geometrical difference between q^0 and σ. But how can we determine the relationship between _′163 and _ 164?

From the principle of maximum entropy, the potential energy function was defined as the negative integral of the force through a distance just as it is in classical mechanics, because

$$F_j = -\frac{dV}{dx^j} \quad ,$$

but the potential energy plays the role of the gauge function. The equations of motion for the asymmetrical forces between unlike forces showed that an analytic form for the potential energy may be unobtainable owing to the transcendental nature of the forces and because the forces depend upon the separation between the particles, not their positions. Thus, there appears no way, at the moment to obtain an analytical expression for _′166, and we must resort to a numerical solution for the unlike particle case.

The absence of an analytical expression for the effective unit of action, $\_'$ 313, does not completely stop us from considering the possibility that a neutron may be a proton in a large orbit about an electron in a small orbit. We may, for the moment, acknowledge the difficulty of obtaining an analytical expression for $\_'$ 314 by allowing the $\_'$ 315, or unit of action, for the proton and electron to be a function of their orbit, and we may designate $\__e$ 316 to be the effective unit of action for the electron orbit in a neutron and $\__p$ 317 to be the unit of action for the neutron's proton orbit. If the effective unit of action depends upon the orbit, as it appears here that it must, then the interpretation that Heisenberg's Uncertainty Principle rules out the possibility of an electron being contained within nuclear discussions is inapplicable.

Another argument against the neutron being an electron and proton in nuclear-sized orbits is based on an argument that the principle of angular momentum cannot be conserved. The neo-coulombic forces, which require that the force between the electron and proton be directed on a line between them, requires that angular momentum be conserved. However, the effective unit of action for the electron orbit requires that, in the neutron the orbital angular momentum would be given by $\__e$ 318 and its intrinsic spin angular momentum would be $\frac{1}{2}\__e$ 319. Similarly, for the proton the orbital angular momentum would be $\__p$ 320 and the spin $(\frac{1}{2})\__p$ 321.

After the neutron decays, the angular momentum is the sum of the two particles' intrinsic spin angular momenta, which is given by $\_$ 322 because both particles are free and, therefore, each has an intrinsic spin angular momentum of $(\frac{1}{2})\_$ 323. Therefore, the conservation of angular momentum is expressed as

$$\frac{1}{2}\left(+\_\_{e}+\_\_{p}\right)+\__e+\__p=\_ \ . \qquad (3.80)$$

Experimental evidence of orbital and/or spin angular momentum is contained in the experimental magnetic moments. If we equate the intrinsic and orbital magnetic moments of the electron and proton while they are in the orbital configuration to the experimental value of the neutron's magnetic moment we have

$$+\_\left(\frac{2}{2}\right)\left(\frac{\_e}{\_}\right)\mu_\beta+\_\left(\frac{2}{2}\right)\left(\frac{\_p}{\_}\right)\mu_n-\left(\frac{\_e}{\_}\right)\mu_\beta+\left(\frac{\_p}{\_}\right)\mu_n=-1.91315\,\mu_n. \qquad (3$$

where μ_β 326 is a Bohr magneton and μ_n 327 is a nuclear magneton.

Equations (3.80) and (3.81) represent two equations in the two unknowns, $\bar{n}_e$ 328 and $\__p$ 329, which may be solved to obtain the effective units of action for the electron and proton orbits making up a neutron such that angular momentum in conserved during neutrons' decay and that the correct magnetic moment of the neutron is ensured. Substituting the experimentally measured values of intrinsic magnetic moments for the electron and proton into Eqn. (3.81) produces a more accurate solution because this contains the anomalous magnetic moments. Then we would have

$$+\_\ 2.002319\left(\frac{\overline{\ }e}{\_}\right)\mu_{\beta}+\_\ 2.79275\left(\frac{\overline{\ }p}{h}\right)\mu_{n}-\left(\frac{\overline{\ }e}{\_}\right)\mu_{p}+\left(\frac{\overline{\ }e}{\_}\right)\mu_{n}=-1.91315\,\mu_{n}. \qquad (3.82)$$

The only simultaneous solution of Eqns. (3.80) and (3.82), for which $\__e$ 331 and $\__p$ 332 are both positive, are

$$\begin{aligned}\__e &= 8.0517 \times 10^{-4}\,\_\ ,\\ \_^p &= 0.66586\,\_\ .\end{aligned} \qquad (3.83)$$

The values of the effective units of action for the proton and electron given in Eqn. (3.83) show that angular momentum is conserved during the decay of a neutron when the neutron is considered to be a proton in orbit around an electron under the neo-coulombic force.

The third major argument against a neutron being a state of electron and a proton orbits stems from the experimental evidence on the violation of Newton's Third Law during decay. That is, the energy of the electron emerging after decay is inconsistent with the equal and opposite columbic forces between an electron and a proton. Here, we find that the neo-coulombic forces are unequal in magnitude and opposite in direction; thus the energy of an electron emerging as the result of crossing from such an orbit cannot be consistent with Newton's third law. There now exists a fourth argument against this picture of a neutron: the possible existence of the neutrino. The above picture of the neutron produces no need to postulate the existence of neutrinos. What then can be said about the experimental evidence that has been put forward in support of the capture of free neutrinos?[12] A conclusive answer will need to await further investigation.

4.13 Nuclear Masses

The difficulty produced by the asymmetry of forces that arises in the interaction of an electron with a proton may be avoided if two protons are considered to be in orbit about the single electron. If we think of a snapshot of such a case we would find that the situation depicted in Fig. 9 allows us to visualize the forces.

Figure 9. Two protons in orbit about a single electron.

The force on the electron would be zero because it has a proton on each side diametrically opposed to one another. The force on each proton will be made up of two parts; one, the force that is due to the presence of the electron, and the other, owing to the other proton. The symmetry guarantees that each proton will experience an identical force, if circular orbits are assumed, toward the center of rotation. The force on the proton on the left would be

$$F_{p_1} = \frac{-|k|}{r^2}\left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)} + \frac{|k|}{(2r)^2}\left[1 - \frac{\lambda_e}{(2r)}\right]e^{-\left(\frac{\lambda_p}{2r}\right)} \tag{3.83}$$

To be sure, quantum mechanical procedure should be used; however, it may be beneficial to begin by assuming circular orbits similar to Bohr's initial approach to atomic structure. This may indicate the potential utility of the force in Eqn. (3.83), as well as perhaps identifying procedures to be used later.

Any nuclear orbits should probably be relativistic; therefore, in cylindrical coordinates, where the velocity for motion in a plane is given by

$$\bar{v} = \dot{r}\hat{r} + r\,\dot{\theta}\,\hat{\theta} \quad ,$$

then we have

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 = \frac{(\dot{r}^2 + r^2\,\dot{\theta}^2)}{c^2}}$$

For circular orbits, this becomes

$$\gamma = \sqrt{1 - \frac{r^2\,\dot{\theta}^2}{c^2}} \tag{3.84}$$

Thus, the relativistic equations of motion for the proton become

$$\frac{d}{dt}\left[\frac{m_p(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})}{\gamma}\right] = \left(\frac{|k|}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right]\dot{r} \qquad (3.8$$

Equation (3.85) separates into two equations

$$\frac{d}{dt}\left[\frac{m_p\,\dot{r}}{\gamma}\right] = \frac{|k|}{r^2}\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right]\hat{r} \quad . \tag{3.86}$$

and

$$\frac{d}{dt}\left[\frac{m_p\,r\dot{\theta}}{\gamma}\right] = 0 \quad .$$

The second of these equations says that the angular momentum is given by

$$\frac{m^p r^2 \dot{\theta}}{\gamma} = L_p = n\_' \quad , \tag{3.87}$$

where $\_'$ 342 indicates that whereas the unit of angular momentum will be a constant for a given orbit, it may be different for different orbits.

The first of Eqn. (3.86) is

$$\frac{d}{dt}\left[\frac{m_p \dot{r}}{\gamma}\right] = \frac{(m_p \ddot{r} - M_p r \dot{\theta}^2)}{\gamma} - \frac{m_p \dot{r}\frac{d\gamma}{dt}}{\gamma^2} = \left(\frac{|k|}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right],$$

but for circular motion $\dot{r} = 0$ 344, therefore,

$$\frac{m_p r \dot{\theta}^2}{\gamma} = \left(\frac{-|k|}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right] \quad . \tag{3.88}$$

Substituting from Eqn. (3.87) into Eqn. (3.88) we have

$$\frac{n^2(\hbar')^2\gamma}{m_p r^3} = \left(\frac{-|k|}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right] \quad . \tag{3.89}$$

The potential energy for one of the protons can be found by integrating the force and is

$$\begin{aligned} V(r) &= -\int . F(r)\, dr = -|k| \int \left(\frac{1}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - \left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right]dr \\ &= \frac{|k|}{r}\left[\left(\frac{1}{4}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - e^{-\left(\frac{\lambda_e}{r}\right)}\right] \quad . \end{aligned} \tag{3.90}$$

Then the total energy of the three-body system, including rest energy, would be

$$E_T = \frac{2|k|}{r}\left[\left(\frac{1}{4}\right)e^{-\left(\frac{\lambda_e}{2r}\right)} - e^{-\left(\frac{\lambda_e}{r}\right)}\right] + \frac{2 m_p c^2}{\gamma} + m_e c^2 \quad . \tag{3.91}$$

However, by substituting Eqn. (3.87) into Eqn. (3.84) and solving for γ, we find

$$\gamma = \frac{1}{\sqrt{1+\left(\frac{n\_'}{m_p r c}\right)^2}} \quad . \tag{3.92}$$

Thus, substituting Eqn. (3.92) into Eqn. (3.89) produces a transcendental equation whose solution gives r(n), which may then be used in Eqn. (3.90) to obtain the total energy of the system. The mass of the system should then be found from

$$M = \left(\frac{E_T}{c^2}\right) \quad . \tag{3.93}$$

Because this system has one electron and two protons, it has a total electric charge of +1 and would have a mass of approximately 2 amu. This is the same characteristic exhibited by the deuterium nucleus. If this is the structure of the H_2 nucleus, then the mass given by Eqn. (3.93) should correspond to the mass of the ground-state nuclear mass for n = 1. If the 1e^-, 2p^+ case existing where n = 1 is the ground-state H_2 nucleus, then is the excited state represented by two protons in the n = 2 state or can it be represented by one proton in an n = 1 orbit and one in an n = 2 orbit? The equations developed here consider only the case when both protons are in the same orbit. Any consideration of the protons being in different orbits introduces an asymmetry in the forces and a similar difficulty faced in the neutron case. Therefore, for the moment we will consider only the simpler cases, where symmetry reduces the complexity of the solution. Notice, though, that even in the simpler symmetric case, no analytical solution exists of Eqn. (3.89) for r(n) because the force contains a transcendental function.

By allowing $\hbar'$ 351to be different for each n, then for the ground and first excited states of the H_2 there are four quantities to be determined: $\lambda_p, \lambda_e, \_'(1)$ 352, and $\_'$ 353. Of course, in theory, we could determine both λ_p 354 and λ_e 355 from scattering experiments: then we would only have two, $\_'(1)$ 356and $\_'(2)$ 357. But if we discover exactly how the $\_'$ 358 depends upon the orbit, then a solution of Eqn. (3.89) would represent a pure theoretical prediction of both the ground state mass, m(n = 1), and the excited state mass, m(n = 2), for then we would be able to express $\_' = \_'(r)$ 359.

If we think about the possibility of adding an additional proton to the 1e^-, 3p^+ case, we are faced with a question. Can the additional proton be placed in the n = 2 orbit without considering the asymmetry thus introduced into the system, or must we consider a single orbit with three protons symmetrically spaced? The answer lies partly in the solution of the appropriate Schrodinger or Dirac equation, because this would inform us of the number of protons that are allowed in a given orbit. This, however, would not answer the question concerning how the asymmetry introduced by a single proton in the n = 2 orbit affects the problem. To obtain an answer to this question the situation should be addressed by both methods and both results should be compared with the experimental evidence.

Whichever approach proves to be correct, the result would be a nucleus with a total charge of +2 and a mass number of 3, or Z = 2, and A = 3. This corresponds to the He_3 nuclei. If the third proton can be placed in the n = 2 orbit of the solution for a single-electron case, and determining λ_p 360 and λ_e 361 by scattering experiments, plus determining _'(1) and _'(2) 362 by the bound and excited states of the H_2 nucleus, then the mass of the He_3 is a pure theoretical prediction. We shall consider this a little later.

The fact that the excited state of the H_2 nucleus is a virtual state implies that adding two protons may itself be nearly a virtual state and thus a cutoff would occur in the number of protons that can exist in a bound state for a one-electron core. To possibly look at other nuclei, suppose first that we go back and consider the like particle force between two electrons,

$$F_{ee} = \frac{|k|}{r^2}\left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)} \tag{3.94}$$

This force is repulsive until the separation between the electrons becomes less than λ_e, and then it becomes attractive. But can there be bound states of two electrons? The answer lies in the effect of the gauge on _' 364. If the _' 365 for unlike particles is less than _' 366, then the sign reversal between the force of unlike particles and the force for like particles should mean that the _' 367 for like particles should be larger than _ 368. Thus, bound states of like particles would be forbidden.

As an example of the sign reversal effect on _' 369, suppose that the gauge for the unlike electron, proton case is given by

$$\ln f^{\frac{1}{2}} = \left(\frac{-k}{r}\right)\left(e^{-\left(\frac{\lambda_p}{r}\right)} - e^{-\left(\frac{\lambda_e}{r}\right)}\right) \ ,$$

or

$$f = \exp\left[\left(\frac{-k}{r}\right)\left(e^{-\left(\frac{\lambda_p}{r}\right)} - e^{-\left(\frac{\lambda_e}{r}\right)}\right)\right] \ . \tag{3.95}$$

Then,

$$(dq^0)^2 = g_{jk}\, dx^j\, d^{xk} = f(d\sigma)^2 f\, \hat{g}_{jk}\, dx^j\, dx^k \ ,$$

or

$$g_{jk} = f\, \hat{g}_{jk} \ .$$

Suppose $\hat{g}_{jk}$ 374 = δ_{jk}375; then Eqn. (3.78) would become

$$\_'\delta_{jk} = \_f\,\hat{g}^{kl}\left[\delta_{jl} + 0\right] = f\_\,\delta_{jk} \quad . \tag{3.96}$$

Then, from Eqn. (3.96), we find

$$\frac{\_'}{\_} = f \quad .$$

Now $\lambda_e << \lambda_p$ 378; therefore, when r = λ_p, the gauge function, f, from Eqn. (3.95) is less than unity because k < 0. Thus the _′379 given by this function is always less than or equal to .

On the other hand, for like particles where only one λ is involved, the _ gauge function would be given by

$$f = \exp\left[\left(\frac{k}{r}\right)e^{-\left(\frac{\lambda}{r}\right)}\right] \quad . \tag{3.97}$$

Then, because k > 0, f ≥ 1, requiring _′ ≥ _ 382.

Although like-particle bound states may be forbidden by the uncertainty principle for large _′383, bound states of unlike particles of nuclear dimensions are allowed by an _′384that may be much, much less than _385. Next, we might consider possible bound states between electrons and positrons with subnuclear dimensions. If the λ for the positron, λ_{e+}, is less than the λ for the electron, λ_{e-}, then we could have bound states with two electrons in orbit about a single positron that would be given by equations exactly like the equations for the $1e^-$, $2p^{2+}$ states, where the positron replaces the electron in the $1e^-$, $2p^+$ equations and the two electrons in the $1e^+$, $2e^-$ case replace the two protons in the $1e^-$, $2p^+$ shell. It is possible now to consider a $1e^+$, $3e^-$ core, thus introducing questions concerning asymmetry aspects and other possible questions. However, owing to the $\lambda_e + < \lambda_e - << \lambda_p$, the core structure is one where in electrons orbit about positrons and the electron orbits are << λ_p. For the protons in a shell orbit the interior core structure may be negligible, just as the internal structure of the nucleus has almost no effect on the atomic electron orbits.

We shall not explore the core structure here but shall consider only the effect of different-core excess electron charge upon allowed proton shell orbits. If we denote the excess electron charge of the core by the integer Y, by which we mean the total number of core electrons less the number or core positrons, then by denoting the number of shell protons in orbit around this nuclear core, we find that the charge on the nucleus, Z, is given by

$$Z = A - Y \tag{3.98}$$

Equation (3.98) indicates that the excess core electron number behaves identically with the neutron numbers in current nuclear theory, although there are no neutrons as such in this nuclear model. Indeed, the neutron, in this picture, is simply another state, namely Y = 1 and A =1.

This suggests a picture of the nucleus in which there are protons in orbits about a nuclear core. The number of protons are given by the current mass number, A. The radii of the proton shell orbits are approximately the value of λ_p; that is, about 1 fermi. The core may be made up of electrons in orbit about positrons and is sized approximately the same as λ_{e-}, which is much, much less than λ_p. This view of the nucleus is similar to that of the atomic view, but here the nuclear core plays the role of the atomic electrons. The force law for the shell proton orbits would then be given, from Eqn. (3.85), by

$$F = \frac{|k|}{r^2}\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - Y\left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right] . \qquad (3.99)$$

The equations specifying the proton shell orbits are

$$\frac{n^2 (h)^2 \gamma}{m_p r^3} = \left(\frac{-|k|}{r^2}\right)\left[\left(\frac{1}{4}\right)\left(1 - \frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} - Y\left(1 - \frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right] \qquad (3.100)$$

and

$$\gamma = \frac{1}{\sqrt{1 + \left(\frac{n\_'}{m_p rc}\right)^2}}$$

The total energy of the nuclei would be given by

$$E(A,Y) = \sum_n \left\{ \frac{A(n)|k|}{R(n)}\left[\frac{1}{4}e^{-\left[\frac{\lambda_p}{2R(n)}\right]} - Ye^{-\left[\frac{\lambda_e}{R(n)}\right]}\right] + \frac{A(n) m_p^2}{\gamma[R(n)}\right\} + E_c(Y) \qquad (3.101)$$

where A(n) is the number of protons with the quantum number, n; R(n) is the radius of the proton orbit with the number n; $E_c(Y)$ is the energy of the nuclear core for which Y is the excess electron charge; and $\gamma[R(n)]$ is the relativistic γ evaluated for R(n). The mass of the nuclei with energies given by Eqn. (3.101) would then be

$$M(A,Y) = \left(\frac{1}{c^2}\right) E(A,Y) \quad . \qquad (3.102)$$

This approach has a simple look to it. For instance, if Y = 1 then E_c = 0.511 MeV, the rest energy of the electron. Then the ground state for ^2H would be

$$E(2,1) = 2\,E_1 + E_c \quad ,$$

whereas the excited state is

$$E(2,1)^* = 2\,E_2 + E_c \quad .$$

Using E_c = 0.511 Me V we find that the energy of a single proton in the n = 2 orbit would be

$$E_2 = \frac{\lfloor E(2,1)^* - E_c \rfloor}{2}$$

Table I. Experimental and predicted nuclear masses.

Y	Z	A	Experimental Mass (MeV)	Predicted Mass (MeV)	ΔM = E_p - E_E (MeV)	ΔM/A (MeV)	BE/A (MeV)	Predicted
1	1	2	1875.0	1873.7	-1.3	-0.7		2.1
1	1	2	1877.9	1879.7	1.8	0.9	1.2	
1	2	3	2808.3	2819.9	2.6	0.9	1.7	
1	3	4	3749.5	3748.2	-1.3	-0.3	1.5	
2	1	3	2808.9	2804.2	-4.7	-1.2	4.4	
2	2	4	3727.3	3736.1	8.8	2.2	4.9	
2	3	5	4667.5	4668.1	-4.8	0.1	5.2	
2	4	6	5604.7	5600.0	-4.7	-0.8	5.4	
3	2	5	4667.8	4668.3	0.5	0.1	5.4	
3	3	6	5601.4	5600	-1.0	-0.2	5.5	
3	4	7	6531.8	6532.3	-1.6	0.1	5.6	
4	1	5	4691.8	4689.2	-1.6	-0.3		1.5
4	2	6	5605.5	5610.4	4.9	0.8		4.1
4	3	7	6533.3	6531.6	-1.6	-0.2		5.9
4	4	8	7454.3	7452.7	-1.6	-0.2		7.3
5	2	7	6545.7	6545.7	0.6	0.1		4.6
5	3	8	7471.2	7471.2	-1.5	-0.2		5.9
5	4	9	8392.2	8395.5	0.8	0.1	--	
6	2	8	7482.5	7482.5	1.0	0.1		3.8
6	3	9	8406.7	8404.7	-2.0	-0.2		5.3
6	4	10	9325.0	9326.0	1.0	0.1		6.5

Thus, the ^{3}He nuclei energy would be

$$E(3,1) = E(2,1) + \frac{\left[E(2,1)^{*} - E_c\right]}{2} \quad .$$

Now using the tabulated experimental data[14] we find that the predicted nuclear mass of the ^{3}He should be

$$E(3,1) = 2.012836 \ amu + \frac{\left[2.016000 - 5.49 \times 10^{-4}\right]}{2} amu$$
$$= 3.020562 \ amu \quad ,$$

compared to the tabulated value of 3.014848 amu. Similarly, the predicted mass for ^{4}Li should then be 4.028288 compared to the tabulated 4.025231. The difference between the predicted and the tabulated values are 1.8 and 0.7 MeV/nuclear, respectively, for the ^{3}He and ^{4}Li nuclei.

Because the core energy and orbital energy levels should change

Table II. Energy-level average values.

Y	$2E_1+E_c$ (amu)	$2E_1+6E_2+E_c$ E_1 (amu)	E_2 (amu)	(amu)
1	2.011441	8.048629	1.002976	1.006198
2	2.009952	8.012742	--	1.000465
3	2.009979	8.013231	--	1.000542
4	2.067271	8.000773	--	0.988917
5	2.071474	8.018938	--	0.991244
6	2.100007	8.032752	--	0.988969

when the excess electron number of the core changes, we may construct Table I, where selected nuclei are used to establish the core and shell energy levels for different Y. Predictions of the mass of other nuclei are made using the energy Eqn. (3.101) and assuming that the number of protons in a full shell corresponds to the number of electrons in the atomic shells, i.e., 2, 8, 18, ... For each Y, some of the experimental masses are used to establish an energy value; therefore, the predicted value appears the same as the experimental. In each case, the energy value established by this data point appears in the appropriate column. The RMS error in the predicted values of all 21 nuclei was 4.3 MeV, with an arbitrary selection of which nuclei were used to establish an energy level.

A better way of approaching the establishment of the energies would be to take the average value of all possible ways to find a particular energy. Table II lists the energy-level average values needed in the total energy equation for the same 21 nuclei. By using the average values from this table in Eqn. (3.101) we may construct another table, which compares the predicted masses with the experimental masses and also tabulates the predicted binding energy per nucleon (i.e., BE/A). The RMS error in the predicted masses in Table III was 2.9 MeV. A comparison between the M/A and BE/A will readily display the predicted error in the binding energy per nucleon, because M/A is the error in mass per nucleon; therefore, the sum of M/A and the predicted BE/A is the experimental binding energy per nucleon.

In the development of the energy, Eqn. (3.101), we assumed that an extra proton could be added in an orbit and the interaction between that new proton and the other protons could be ignored.

This assumption was made even after we saw that any odd proton sets up asymmetrical forces. Thus, errors could have been expected. Still, the RMS errors from this crude averaging procedure do not appear too inaccurate when even the best of the semiempirical nuclear mass formulas does not address nuclei below a Z of 16 because of the large errors that arise.[15]

To avoid the errors resulting from ignoring proton-proton interaction, we must reconsider the simplest case, a single electron and three protons in orbit around this electron. Proton-proton interaction suggests that the protons will arrange themselves in a plane spaced on the points of an equilateral triangle, as in Fig. 10. For this case, the force on a single proton would be

$$F_3 = \frac{-|k|Y}{r^2}\left(1-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)} + \frac{2|k|}{R^2}\cos\left(\frac{\pi}{6}\right)\left(1-\frac{\lambda_p}{r}\right)e^{-\left(\frac{\lambda_p}{R}\right)} \ ,$$

where R = $\sqrt{3r}$ 397. Then the force becomes

$$F_3 = \frac{|k|}{r^2}\left[\left(\frac{1}{3}\right)\left(1-\frac{\lambda_e}{\sqrt{3r}}\right)e^{-\left(\frac{\lambda_e}{3r}\right)} - Y\left(1-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right] \ . \tag{3.103}$$

This force differs from the two-proton force of Eqn. (3.99) by the coefficient of the separation, r, in the proton-proton portion of the force. The relativistic circular orbit equations of motion would be

$$\frac{n^2(\overline{n'})^2\gamma}{m_p r^3} = \left(\frac{-|k|}{r^2}\right)\left(\frac{1}{3}\right)\left(1-\frac{\lambda_p}{\sqrt{3r}}\right)e^{-\left(\frac{\lambda_p}{r}\right)} - Y\left(1-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)} \ , \qquad (3.10$$

which differs from Eqn. (3.100) only be replacing R = 2r by R = $\sqrt{3r}$ 400. Then the energy is

$$E_3(n,Y)=\left\{\frac{3|k|}{R(n)}\left[\left(\frac{1}{3}\right)e^{-\left(\frac{\lambda_r}{\sqrt{3r}}\right)}-Ye^{-\left(\frac{\lambda_e}{r}\right)}\right]+\frac{3m_pc^2}{\gamma[R(n)]}\right\}+E_c4 \quad . \qquad (3.105)$$

Table III. Mass predictions for selected nuclei.

Y	Z	A	Expermental Mass (amu)	$2E_1$ + Ec (amu)	E_2 (amu)	Predicted Mass (amu)	♠M E_P - E_E (MeV)/A
1	1	2	2.012836	2.012836(2)	-	2.012836	0
1	1	2	2.016000	Excited ^{2}H	1.00773(2)	2.016000	0
1	2	3	3.014848	(2)	1.001484(1)	3.020562	1.8
1	3	4	4.025231	(2)	1.010179(2)	4.028288	
2	1	3	3.015484	(2)	(1)	2.998268	-5.3
2	2	4	4.001422	(2)	(2)	4.004472	0.7
2	3	5	5.010676	1.992064(2)	(3)	5.010676	0
				(2)			
2	4	6	6.016880	1.006204(4)		6.016880	0
3	2	5	5.011046	2.004404(2)	(3)	5.011046	0
3	3	6	6.013260	(2)	1.002214(4)	6.013260	0
3	4	7	7.012129	(2)	(5)	7.015474	0.4
4	1	5	5.035709	(2)	(3)	5.036203	0.1
4	2	6	6.017709	(2)	(4)	6.024955	1.1
4	3	7	7.013707	2.069947(2)	(5)	7.013707	0
4	4	8	8.002459	2.056639(2)	0.988752(6)	8.002459	0
5	2	7	7.026850	2.057980(2)	(5)	7.026850	0
5	3	8	8.020624	(2)	0.993774(6)	8.020624	0
5	4	9	9.009337	(2)	(7)	9.014398	0.5
6	2	8	8.032752	2.079702(2)	(6)	8.032752	0
6	3	9	9.024927	0.992175(7)		9.024245	-0.1
6	4	10	10.010689	(2)	(8)	10.017102	0.6

If we consider a possible symmetric orbit of four protons, as pictured in Fig. 11, then the force on each proton would be

$$F_4=\frac{-Y|k|}{r^2}1\left(-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}+\frac{2|k|\cos\left(\frac{\pi}{4}\right)}{R^2}\left(1-\frac{\lambda_p}{R}\right)e^{-\left(\frac{\lambda_p}{r}\right)}+\frac{|k|}{(2r)^2}\left(1-\frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)} \quad . \qquad (3.106)$$

where R = $\sqrt{2r}$ 402.

Figure 10. Four protons in symmetric orbit.

Thus Eqn. (3.106) becomes

$$F^4 = \frac{-|\bar{k}|}{r^2}\left[\frac{\sqrt{2}}{2}\left(1-\frac{\lambda_p}{\sqrt{2}r}\right)e^{-\left(\frac{\lambda_p}{\sqrt{2}r}\right)}+\left(\frac{1}{4}\right)\left(1-\frac{\lambda_p}{2r}\right)e^{-\left(\frac{\lambda_p}{2r}\right)}-Y\left(1-\frac{\lambda_e}{r}\right)e^{-\left(\frac{\lambda_e}{r}\right)}\right]$$
$$=\frac{\sqrt{2}\,|k|}{2r^2}1-\left(\frac{\lambda_p}{\sqrt{2}r}\right)e^{-\left(\frac{\lambda_p}{\sqrt{2}r}\right)}+F_2(Y)\ , \tag{3.107}$$

where F_2 is the two-proton force of Eqn. (3.99). We can also write the circular motion equation of motion as

$$\frac{n^2(h')^2\gamma}{m_p r^3}=\frac{-\sqrt{2}\,|k|}{2r^2}\left(1-\frac{\lambda_p}{\sqrt{3}r}\right)e^{-\left(\frac{\lambda_p}{\sqrt{3}r}\right)}-F_2(Y)\ , \tag{3.108}$$

and the energy equation as

$$E_4(n,Y)=\frac{4|k|}{R(n)}\left[\left(\frac{\sqrt{2}}{2}\right)e^{-\left(\frac{\lambda_p}{\sqrt{2}r}\right)}+\left(\frac{1}{4}\right)e^{-\left(\frac{\lambda_p}{2r}\right)}+Ye^{-\left(\frac{\lambda_{e^-}}{r}\right)}\right]+\frac{4m_pc^2}{\gamma[R(n)]}+E_c(Y)$$
$$=\frac{2\sqrt{2}\,|k|}{R(n)}e^{-\left(\frac{\lambda_p}{\sqrt{2}r}\right)}+2E_2(n,y)-E_c(Y)\ . \tag{3.109}$$

The foregoing discussion on a possible method of accounting for proton-proton orbit interaction seems to imply that we must do every possible nuclear configuration differently. The final answer to this question must await further research. The discussion was presented to point out that ignoring the proton-proton interactions may be an error source in the predicted masses. For instance, consider the force for the symmetric four-proton orbit, F_4, given by Eqn. (3.107). This force may be a stronger attractive force for a given Y and r than F_2, so long as $r<\lambda_p/\sqrt{2}$ 406, but it is weaker for $r>\lambda_p/\sqrt{2}$ 407. Thus, for orbits where $r<\lambda_p/\sqrt{2}$ 408, four protons with n = 1 can produce less ground-state energy from E_4 than the energy expression in Eqn. (3.99) does with two protons in the n = 1 orbit and two in the n = 2 orbit. The large error would thus diminish in the prediction of the ^4He nuclei mass.

The appearance of the exponential term requires the use of numerical solutions, whether one tries the circular orbit approximation or uses the quantum mechanical approach. This will be the subject of future research. However, the masses predicted by the very simple assumptions and the comparison of experimental and predicted binding energy per mass number, plotted in Fig. 7, seem to imply that a more detailed solution may prove very useful.

Figure 11. Binding energy per mass number versus mass number.