Chapter 7 Hydrodynamic Systems

The equation of motion for the fifth dimension, mass density, appears as a generalization of the principle of the conservation of mass. Further, in classical hydrodynamic systems five equations in five unknowns are used. It seems logical then to expect the five equations of motion appearing in the five-dimensional Dynamic Theory to be generalizations of the classical equations. An added incentive to investigate the possibilities of this generalization is gained when electromagnetically contained ionized plasmas with mass conversion are considered. For if the five equations are generalizations of the classical hydrodynamic equations, then the use of the five-dimensional fields allowing mass conversion should provide an entirely new viewpoint of a controlled fusion reactor.

Since it is suspected that the five equations of motion resulting from the application of the principle of increasing entropy to a thermo-mechanical system are generalizations of the classical equations, it then becomes necessary to show that this is indeed the case. This seems possible by restricting the system so that it corresponds to the usual system considered.

First, from the Dynamic Theory approach, the manifold required for a description of the system is the five-dimensional manifold of space, time, and mass density. Within this manifold the continuity equation of mass no longer holds for the general system. We can, however, restrict our system by first requiring that the system remain on a hypersurface within the five-dimensional manifold. For a system so restricted, any of the five dimensions may be considered as functions of the other four. In particular, since by custom in hydrodynamics the mass density is considered to be a function of space and time, we may consider the mass density to be the variable chosen to be function of the others or

$$\gamma = \gamma\left(x^0, x^1, x^2, x^3\right)$$

so that

$$d\gamma = \left(\frac{\partial \gamma}{\partial x^\alpha}\right) dx^\alpha \quad .$$

Such a system will be constrained to be on a hypersurface embedded within the five-dimensional manifold of space, time, and mass density as shown and upon this hypersurface will be described in a four-dimensional manifold of space and time.

If we further restrict our system by requiring that the total derivative of the mass density to be zero or

$$d\gamma = 0 = \frac{d\gamma}{dx^{\alpha}} dx^{\alpha}$$

then

$$\frac{d\gamma}{dt} = 0 = \frac{\partial\gamma}{\partial t} + \frac{\partial\gamma}{\partial x^{1}} v^{1} + \frac{\partial\gamma}{\partial x^{2}} v^{2} + \frac{\partial\gamma}{dx^{3}} v^{3}$$

or

$$\frac{\partial\gamma}{\partial t} + grad\,\gamma \bullet \bar{v} = 0 \quad ,$$

which is the usual continuity equation. Thus, by restricting the system to this particular hypersurface we have constrained the system to obey the continuity equation as does a usual hydrodynamic system.

Not only does this restriction place our system within the space-time manifold where we may compare the resulting four equations of motion with the equations of motion in relativistic theories but, since the seven gauge field equations must hold in the five-dimensional manifold they must also hold on the hypersurface. This allows the new field quantities to be expressed as functions of the □, B fields and the partial derivatives of the mass densities. Further, it appears that the additional B field equations may be used to determine a dependence of the E and B fields upon the mass density and/or its changes.

Then by comparing the equations of motion obtained here for the system restricted to the mass conservation hypersurface with the relativistic Navier-Stokes equations it should be possible to identify the viscous coefficients with the field quantities and perhaps see how the viscosity depends upon these fields as I feel it does.

Since we have restricted the system to a hypersurface where the mass density is a function of space and time, then the surface is defined by five equations of the type

$$x^{i} = x^{i}(u^{0}, u^{1}, u^{2}, u^{3}) \quad . \tag{7.1}$$

Further, since $x^4 = \gamma/a_0$ and $x^4 = x^4(x^0, x^1, x^2, x^3)$, then Eqn. (7.1) becomes

$$x^{0} = u^{0}, x^{1} = u^{1}, x^{2} = u^{2}, x^{3} = u^{3}$$

and

$$x^{4} = f(u^{0}, u^{1}, u^{2}, u^{3}) \quad .$$

Since u^0, u^1, u^2, and u^3 are independent variables, the locus defined by Eqn. (7.1) is four-dimensional, and these equations give the coordinates x^i of a point on the hypersurface when u^0, u^1, u^2, and u^3 are assigned particular values. This point of view leads one to consider the surface as a four-dimensional manifold S embedded in a five-dimensional enveloping space. We can also study surfaces without reference to the surrounding space, and consider parameters u^0, u^1, u^2, and u^3 as coordinates of points in the surface.

If we assign to u^0 in Eqn. (7.1) some fixed value $u^0 = u^0$, we obtain a three-dimensional manifold

$$x^i = x^i(u^0, u^1, u^2, u^3), (i = 0, 1, 2, 3, 4)$$

which is a three-dimensional manifold lying on the hypersurface S defined by Eqn. (7.1). By assigning fixed values for any three of the four hypersurface variables we obtain a net of curves, on the hypersurface, which may be called coordinate curves.

Obviously the parametric representation of a hypersurface in the form of Eqn. (7.1) is not unique, and there are infinitely many curvilinear coordinate systems which can be used to locate points on a given hypersurface S. Thus, if one introduces a transformation

$$u^0 = u^0(u^{-0}, u^{-1}, u^{-2}, u^{-3}) \quad ,$$
$$u^1 = u^1(u^{-0}, u^{-1}, u^{-2}, u^{-3}) \quad ,$$
$$u^2 = u^2(u^{-0}, u^{-1}, u^{-2}, u^{-3}) \quad ,$$

and

$$u_3 = u^3(u^{-0}, u^{-1}, u^{-2}, u^{-3}) \quad ,$$

(7.2)

where the u^α (u^{-0}, u^{-1}, u^{-2}, u^{-3}) are of class C^1 and are such that the Jacobian

$$J = \frac{\partial(u^O, u^1, u^2, u^3)}{\partial(u^{-0}, u^{-1}, u^{-2}, u^{-3})}$$

does not vanish in some region of the variables u, then one can insert the values from Eqn. (7.2) in Eqn. (7.1) and obtain a different set of parametric equations

$$x^i = f^i(u^{-0}, u^{-1}, u^{-2}, u^{-3})$$

(7.3)

defining the hypersurface S. Equation (7.2) can be looked upon as representing a transformation of coordinates in the hypersurface.

7.1 First Fundamental Quadratic Form

The properties of hypersurfaces that can be described without reference to the space in which the hypersurface is embedded are termed "intrinsic" properties. A study of intrinsic properties is made to depend on a certain quadratic differential form describing the metric character of the hypersurface. We proceed to derive this quadratic form for our restricted system.

It will be convenient to adopt certain conventions concerning the meaning of indices to be used. We will be dealing with two distinct sets of variables: those referring to the five-dimensional space in which the hypersurface is embedded (these are five in number) and with four coordinates u^0, u^1, u^2, and u^3 referring to the four-dimensional manifold S. In order not to confuse these sets of variables we shall use Latin letters for the indices referring to the space variables and Greek letters for the hypersurface variables. Thus, Latin indices will assume values 0, 1, 2, 3, 4 and Greek indices will have the range of values 0, 1, 2, 3. A transformation T of space coordinates from one system $\underline{X}$ to another $\underline{X}$ will be written as

a transformation of Gaussian hypersurface coordinates, such as described by Eqn. (7.2) will be denoted by

A repeated Greek index in any term denotes the summation from 0 to 3; a repeated Latin index represents the sum from 0 to 4. Unless a statement to the contrary is made, we shall suppose that all functions appearing in the discussion are of class C^2 in the regions of their definitions.

Consider the hypersurface S defined by

$$x^i = x^i(u^0, u^1, u^2, u^3) \quad , \tag{7.4}$$

where the x^i are coordinates covering the five-dimensional space in which the hypersurface S is embedded, and a curve C on S defined by

$$u_\alpha = u^\alpha(\tau) \quad , \quad \tau_1 \le \tau \le \tau_2 \tag{7.5}$$

where the u^α's are the Gaussian coordinates covering S. Viewed from the surrounding space, the curve defined by Eqn. (7.4) is a curve in a five-dimensional manifold, which we shall assume, for the present, is

Riemannian entropy manifold of the Dynamic Theory, and its element of arc is given by the formula

$$(dq^0)^2 = g_{ij}\, dx^i\, dx^j$$

(7.6)

From Eqn. (7.4) we have

$$dx^i = \frac{\partial x^i}{\partial u^\alpha}\, du^\alpha$$

(7.7)

where, as is clear from (7.5),

$$du^\alpha = \frac{du^\alpha}{d\tau}\, d\tau \quad .$$

Substituting from Eqn. (7.6) and Eqn. (7.7), we get

$$\left(dq^0\right)^2 = \hat{g}_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}\, du^\alpha\, du^\beta$$
$$= A_{\alpha\beta}\, du^\alpha\, du^\beta \quad ,$$

where

$$A_{\alpha\beta} \equiv \hat{g}_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \quad .$$

(7.8)

The expression for $(dq^0)^2$, namely

$$\left(dq^0\right)^2 = A_{\alpha\beta}\, du^\alpha\, du^\beta \quad ,$$

is the square of the linear element of C lying on the hypersurface S, and the right hand member of (7.8) can be called the First Fundamental quadratic form of the hypersurface. The length of arc of the curve is given by

$$q_2^0 - q_1^0 = \int_{\tau_1}^{\tau_2} \gamma \sqrt{_{-} A_{\alpha\beta}\, \dot{u}^\alpha\, \dot{b}^\beta}\; d\tau \quad ,$$

where
$\dot{u}^\alpha = \frac{du^\alpha}{d\tau}$ 25 and q^0 is the specific entropy. The total change in the entropy along the curve C would then be

$$\gamma\left(q_2^0 - q_1^0\right) = \int_{\tau_1}^{\tau_2} \gamma \sqrt{-A_{\alpha\beta}\, u^\alpha u^\beta}\, d\tau \quad .$$

(7.10)

Consider a transformation of surface coordinates

$$u^\alpha = u^\alpha(\bar{u}^0, \bar{u}^1, \bar{u}^2, \bar{u}^3)$$

(7.11)

with a non-vanishing Jacobian

$$J = \left| \frac{\partial u^\alpha}{\partial \bar{u}^\beta} \right|$$

It follows from Eqn. (7.11) that

$$du^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} d\bar{u}^\beta \quad ,$$

and hence Eqn. (7.9) yields

$$\left(dq^0\right)^2 = A_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\delta} d\bar{u}^\gamma d\bar{u}^\delta \quad .$$

If we set

$$A_{\gamma\delta} = A_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\gamma} \frac{\partial u^\beta}{\partial \bar{u}^\delta} \quad ,$$

we see that the set of quantities $A_{\alpha\beta}$ represents a symmetric covariant tensor of rank two with respect to the admissible transformations Eqn. (7.11) of hypersurface coordinates. The fact that the $A_{\alpha\beta}$ are components of a tensor is also evident from Eqn. (7.9), since $(dq^0)^2$ is an invariant and the quantities $A_{\alpha\beta}$ are symmetric. The tensor $A_{\alpha\beta}$ is called the covariant metric tensor of the hypersurface.

Since the form Eqn. (7.9) is positive definite, the determinant

$$A = \left| A_{\alpha\beta} \right| > 0$$

and we can define the reciprocal tensor $A^{\alpha\beta}$ by the formula $A^{\alpha\beta} A_{\beta\gamma} =$

The properties of surfaces concerning the study of the first fundamental quadratic form

$$\left(dq^0\right)^2 = A_{\alpha\beta}\, du^\alpha du^\beta$$

constitute a body of what is known as the 'intrinsic geometry of surfaces.' They take no account of the distinguishing characteristics of surfaces as they might appear to observer located in the surrounding space. Two surfaces, a cylinder and a cone, for example, appear to be entirely different when viewed from the enveloping space, and yet their intrinsic geometries are completely indistinguishable since the metric properties of cylinders and cones can be described by the identical expressions for square of the element of arc. If a coordinate system exists on each of the two surfaces such that the linear elements on them are characterized by the same metric coefficients $A_{\alpha\beta}$, the surfaces are called "isometric."

Thus, if our description of the restricted system is done only in terms of the intrinsic geometry of the hypersurface we may lose sight of features which may characterize our system when viewed from the enveloping space. Therefore, in order to characterize the shape of the surface we must develop a view which involves the enveloping space.

7.2 Second Fundamental Quadratic Form

An entity that provides a characteristic of the shape of the surface as it appears from the enveloping space is the normal line to the surface. The behavior of the normal line as its foot is displaced along the surface depends on the shape of the surface, and it occurred to Gauss to describe certain properties of surfaces with the aid of a quadratic form that depends in a fundamental way on the behavior of the normal line. Before we introduce this new quadratic form let us recall the definition Eqn. (7.8),

$$A_{\alpha\beta} \equiv \hat{g}_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} (i, j = 0, 1, 2, 3, 4) (\alpha, \beta = 0, 1, 2, 3) \quad .$$

We note that the foregoing formulas depend on both the Latin and Greek indices, and we recall that the Latin indices run from 0 to 4 and refer to the surrounding space, whereas the Greek indices assume values 0, 1, 2, and 3 and are associated with the embedded hypersurface. Furthermore, the dx^i and g_{ij}'s are tensors with respect to the transformations induced on the space variables x^i, whereas such quantities du^α and $A_{\alpha\beta}$ are tensors with respect to the transformation of Gaussian surface coordinates u^α. Equation (7.8) is a curious one since it contains partial derivatives

$\frac{\partial x^i}{\partial u^\alpha}$ 35 depending on both Latin and Greek indices. Since both $A_{\alpha\beta}$ and g_{ij} in Eqn (7.8) are tensors, this formula suggests that

$\frac{\partial x^i}{\partial u^\alpha}$ 36 can be regarded either as a contravariant space vector or as a covariant surface vector. Let us investigate this set of quantities more closely.

Let us take a small displacement on the hypersurface S, specified by the surface vector du^{α}. The same displacement, as is clear from Eqn. (7.7), is described by the space vector with components

$$dx^i = \frac{\partial x^i}{\partial u^{\alpha}} du^{\alpha} \quad .$$

(7.12)

The left-hand member of this expression is independent of the Greek indices, and hence it is invariant relative to a change of the surface coordinates u^{α}. Since du^{α} is an arbitrary surface vector, we conclude that

$$\frac{\partial x^i}{\partial u^{\alpha}}$$

(7.13)

is a covariant surface vector. On the other hand, if we change the space coordinates, the du^{α}, being a surface vector, is invariant relative to this change, so the Eqn. (7.13) must be a contravariant space vector. Hence we can write Eqn. (7.13) as

$$x^i_{\alpha} \equiv \frac{\partial x^i}{\partial u^{\alpha}}$$

(7.14)

where the indices properly describe the tensor character of this set of quantities.

Let A and B be a pair of surface vectors drawn from one point P of S.

FIG HERE

Then using Eqn. (7.14) they can be represented in the form

$$A^i = x^i_{\alpha} A^{\alpha} \; and \; B^i = x^i_{\alpha} B^{\alpha}$$

(7.15)

The five-dimensional vector product, defined by

$$N^k = \varepsilon^{kij} A_i B_j \quad ,$$

(7.16)

is the vector normal to the tangent plane determined by the vectors A and B, and the unit vector n perpendicular to the tangent plane, so oriented that A, B, and n form a right-handed system, is

$$\bar{n} = \frac{\varepsilon^{kij} A_i B_j}{\left| \varepsilon^{\alpha\beta} A_\alpha B_\beta \right|}$$

(7.17)

We call the vector n the unit normal vector to the hypersurface S at P. Clearly, n is a function of coordinates (u^0, u^1, u^2, u^3), and as the point P(u^0, u^1, u^2, u^3) is displaced to a new position P($u^0 + du^0$, $u^1 + du^1$, $u^2 + du^2$, $u^3 + du^3$), the vector n undergoes a change

$$d\bar{n} = \frac{\partial \bar{n}}{\partial u^\alpha} du^\alpha$$

(7.18)

whereas the position vector r is changed by the amount

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u^\alpha} du^\alpha \quad .$$

Let us form the scalar product

$$d\bar{n} \bullet d\bar{r} = \frac{\partial \bar{n}}{\partial u^\alpha} \bullet \frac{\partial \bar{r}}{\partial u^\beta} du^\alpha du^\beta \quad .$$

(7.19)

If we define

$$b_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \bar{n}}{\partial u^\alpha} \bullet \frac{\partial \bar{r}}{\partial u^\beta} + \frac{\partial \bar{n}}{\partial u^\beta} \bullet \frac{\partial \bar{r}}{\partial u^\alpha} \right)$$

so that Eqn. (7.19) reads

$$d\bar{n} \bullet d\bar{r} = - b_{\alpha\beta} du^\alpha du^\beta \quad ,$$

(6.20)

the left-hand member of Eqn. (7.20), being the scalar product of two vectors in a Riemannian space by being in the entropy manifold, is an invariant; moreover, from symmetry with respect to α and β , it is clear that the coefficients $du^\alpha du^\beta$ in the right-hand member of Eqn. (7.20) define a covariant tensor of rank two. The quadratic form

$$B \equiv b_{\alpha\beta}\, du^{\alpha}\, du^{\beta} \quad ,$$

(7.21)

called the second fundamental quadratic form of the hypersurface, will be shown to play an essential part in the study of hypersurfaces when they are viewed from the surrounding space, just as the first fundamental quadratic form
$A \equiv d\bar{r} \bullet d\bar{r}$ 49 or

$$A = A_{\alpha\beta}\, du^{\alpha}\, du^{\beta} \quad ,$$

did in the study of intrinsic properties of a hypersurface.

We can rewrite the formula Eqn. (7.17) in terms of the components x_{α}^{i} of the base vectors a_{α}. We denote the covariant components of n by n^{i} and observe that its covariant components n_i are given by

$$n_i = \frac{\varepsilon_{ijk}\, A^{j}\, B^{k}}{\overline{A}\overline{B}\sin\theta}$$

(7.22)

and

$$A\, B \sin\theta = \varepsilon_{\alpha\beta}\, A^{\alpha}\, A^{\beta} \quad .$$

(7.23)

Substituting in Eqn. (7.22) from Eqn. (7.15) and Eqn. (7.23), we get

$$\left(n_i\, \varepsilon_{\alpha\beta} - \varepsilon_{ijk}\, x_{\alpha}^{j}\, x_{\beta}^{k}\right) A^{\alpha}\, B^{\beta} = 0$$

and, since this relation is valid for all surface vectors, we conclude that

$$n_i\, \varepsilon_{\alpha\beta} = \varepsilon_{ijk}\, x_{\alpha}^{j}\, x_{\beta}^{k} \quad .$$

(7.24)

Multiplying Eqn. (7.24) by $\varepsilon^{\alpha\beta}$, and noting that $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta}$=2, we get the desired result

$$n_i = \frac{1}{2}\varepsilon^{\alpha\beta}\, \varepsilon_{ijk}\, x_{\alpha}^{j}\, x_{\beta}^{k} \quad .$$

(7.25)

It is clear from the structure of this formula that n_i is a space vector which does not depend on the choice of surface coordinates. This fact is also obvious from purely geometric considerations.

7.3. Tensor Derivatives

We wish to reduce the second fundamental quadratic form eqn. (7.21) analytically by the operation of tensor differentiation of tensor fields which are functions of both surface and space coordinates. To do this we shall first present the concept of tensor differentiation introduced by A. J. McConnell*.

Let us consider a curve C lying on a given hypersurface S and a vector A^i defined along C. If τ is a parameter along C, we can compute the intrinsic derivative

$\frac{\delta A^i}{\delta \tau}$ 56 of A^i, namely,

$$\frac{\delta A^i}{\delta \tau} = \frac{dA^i}{dt} + \hat{g}\begin{Bmatrix} i \\ jk \end{Bmatrix} A^j \frac{dx^k}{d\tau} \quad , \tag{7.26}$$

In formula eqn. (7.26) the Christoffel symbols

$\hat{g}\begin{Bmatrix} i \\ jk \end{Bmatrix}$ 58 refer to the space coordinates x^i and are formed from the metric coefficients g_{ij}. This is indicated by the prefix

$\hat{g}$ 59 on the symbol. On the other hand, if we consider a surface vector A defined along the same curve C, we can form the intrinsic derivative with respect to the surface variables, namely,

$$\frac{\delta S^\alpha}{\delta \tau} = \frac{dA^\alpha}{d\tau} + a\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} A^\beta \frac{du^\gamma}{d\tau} \quad . \tag{7.27}$$

In this expression the Christoffel symbols

$a\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}$ 61 are formed from the metric coefficients $a_{\alpha\beta}$ associated with the Gaussian hypersurface coordinates u_α. A geometric interpretation of these formulas is at hand when the fields A^i and A^α are such that

$\frac{\delta A^i}{\delta \tau} = 0$ 62 and

$\frac{\delta A^\alpha}{\delta \tau} = 0$ 63. In the first equation the vectors A^i form a parallel field with respect to C, considered as a space curve, whereas the equation

$\frac{\delta A^{\alpha}}{\delta\tau}=0$ 64 defines a parallel field with respect to C regarded as a surface curve. The corresponding formulas for the intrinsic derivatives of the covariant vectors A_i and A_α are

$$\frac{\delta A_i}{\delta\tau}=\frac{d A_i}{d\tau}-\hat{g}\begin{Bmatrix} k \\ ij \end{Bmatrix} A_k \frac{dx^j}{d\tau}$$

(7.28)

and

$$\frac{\delta A_\alpha}{\delta\tau}=\frac{d A_\alpha}{d\tau}-a\begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} A_\gamma \frac{du^\beta}{d\tau} \quad .$$

(7.29)

Consider next a tensor field T^i_α 67, which is a contravariant vector with respect to a transformation of space coordinate x^i and a covariant vector relative to a transformation of surface coordinates u^α . An example of a field of this type is the tensor $x^I_\alpha=\frac{\partial x^i}{\partial n^\alpha}$ 68 introduced earlier. If T^i_α 69 is defined over a surface curve C, and the parameter along C is τ, then T^i_α 70 is a function of τ. We introduce a parallel vector field A_i along C, regarded as a space curve, and a parallel vector field B^α along C, viewed as a surface curve, and form an invariant

$$\phi(\tau)=T^i_\alpha A_i B^\alpha \quad .$$

The derivative of $\varphi(\tau)$ with respect to the parameter τ is given by the expression

$$\frac{d\phi}{d\tau}=\frac{dT^i_\alpha}{d\tau}A_i B^\alpha+T^i_\alpha\frac{dA^i}{d\tau}B^\alpha+T^i_\alpha A_i\frac{dB^\alpha}{d\tau} \quad ,$$

(7.30)

which is obviously an invariant relative to both the space and surface coordinates. But, since the fields $A_i(\tau)$ and $B^\alpha(\tau)$ are parallel,

$$\frac{d A_i}{d\tau}=\hat{g}\begin{Bmatrix} k \\ ij \end{Bmatrix} A_k\frac{dx^j}{d\tau} \text{ and } \frac{dB^\alpha}{d\tau}=\bar{a}\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} B^\beta\frac{du^\gamma}{d\tau} \quad ,$$

and eqn. (7.30) becomes

$$\frac{d\phi}{d\tau} = \left[\frac{dT^i_\alpha}{d\tau} + \hat{g} \begin{Bmatrix} i \\ jk \end{Bmatrix} T^j_\alpha \frac{dx^k}{d\tau} - \bar{a} \begin{Bmatrix} \delta \\ \beta\gamma \end{Bmatrix} T^i_\delta \frac{du^\gamma}{d\tau} \right] A_i B^\alpha \quad .$$

(7.31)

Since this is invariant for an arbitrary choice of parallel fields A_i and B^α, the quotient law guarantees that the expression in the brackets of Eqn. (7.31) is a tensor of the same character as
T^i_α 75. We call this tensor the intrinsic tensor derivative of
T^i_α 76 with respect to the parameter τ, and write

$$\frac{\delta\, T^i_\alpha}{\delta t} = \frac{d\, T^i_\alpha}{d\tau} + \hat{g} \begin{Bmatrix} i \\ jk \end{Bmatrix} T^j_\alpha \frac{dx^k}{d\tau} = a \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} T^i_\delta \frac{du^\gamma}{d\tau} \quad .$$

If the field
T^i_α 78 is defined over the entire hypersurface S, we can argue that, since

$$\frac{\delta\, T^i_\alpha}{\delta\tau} \equiv \left[\frac{\partial T^i_\alpha}{\partial u^\gamma} + g \begin{Bmatrix} i \\ jk \end{Bmatrix} T^j_\alpha\, x^k_\gamma - a \begin{Bmatrix} \delta \\ \gamma\alpha \end{Bmatrix} T^i_\delta \right] \frac{du^\gamma}{d\tau}$$

is a tensor field and
$\frac{du^\gamma}{d\tau}$ 80 is an arbitrary surface vector (for C is arbitrary), the expression in the bracket is a tensor of the type
$T^i_{\alpha\gamma}$ 81. We write

$$T^i_{\alpha\gamma} \equiv \frac{\partial T^i_\alpha}{\partial u^\gamma} + \hat{g} \begin{Bmatrix} i \\ jk \end{Bmatrix} T^j_\alpha\, x^k_\gamma - a \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} T^i_\delta$$

(7.32)

and call
$T^i_{\alpha,\gamma}$ 83 the tensor derivative of
T^i_α 84 with respect to u^γ.

The extension of this definition to more complicated tensors is obvious from the structure of Eqn. (7.32). Thus the tensor derivative of
$T^i_{\alpha\beta}$ 85 with respect to u^γ is given by

$$T^i_{\alpha\beta,\gamma} = \frac{\partial T^i_{\alpha\beta}}{\partial u^\gamma} + g \begin{Bmatrix} i \\ jk \end{Bmatrix} T^i_{\alpha\beta}\, x^k_\gamma - a \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} T^i_{\delta\beta} - a \begin{Bmatrix} \delta \\ \beta\gamma \end{Bmatrix} T^i_{\alpha\delta} \quad .$$

(7.33)

If the surface coordinates at any point P or S are geodesic, and the space coordinates are orthogonal Cartesian, we see that at that point the tensor derivatives reduce to the ordinary derivatives. This leads us to conclude that the operations of tensor differentiations of products and sums follow the usual rules and that the tensor derivatives of g_{ij}, $A_{\alpha\beta}$, G_{ijk}, $\varepsilon_{\alpha\beta}$ and their associated tensors vanish. Accordingly, they behave as constants in the tensor differentiation.

The apparatus developed in the preceding section permits us to obtain easily and in the most general form an important set of formulas due to Gauss. We will also deduce with its aid the second fundamental quadratic form of a surface already encountered.

We begin by calculating the tensor derivative of the tensor x^i_α 87, representing the components of the surface base vectors a_α. We have

$$x^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \hat{g}\{ {i \atop jk} \} x^j_\alpha x^k_\beta - a\{ {\delta \atop \alpha\beta} \} x^i_\delta \quad ,$$

from which we deduce that

$$x^i_{\alpha,\beta} = x^i_{\alpha,\beta} \quad .$$

(7.34)

Since the tensor derivative of $a_{\alpha\beta}$ vanishes, we obtain, upon differentiating the relation

$$A_{\alpha\beta} = \hat{g}_{ij}\, x^i_\alpha\, x^j_\beta \quad ,$$

$$\hat{g}_{ij}\, x^i_{\alpha,\gamma}\, x^j_\beta + g_{ij}\, x^i_\alpha\, x^j_{\beta,\gamma} = 0 \quad .$$

(7.35)

Interchanging α, β, γ cyclically leads to two formulas:

$$\hat{g}_{ij}\, x^i_{\beta,\alpha}\, x^j_\gamma + \hat{g}_{ij}\, x^i_\beta\, x^j_{\gamma,\alpha} = 0$$

(7.36)

and

$$\hat{g}_{ij}\, x^i_{\gamma,\beta}\, x^j_\alpha + \hat{g}_{ij}\, x^i_\gamma\, x^J_{\alpha,\beta} = 0 \quad .$$

(7.37)

If we add Eqn. (7.36) and Eqn. (7.37), subtract Eqn. (7.35), and take into account the symmetry relation Eqn. (7.34), we obtain

$$g_{ij}\, x^i_{\alpha,\beta}\, x^j_\gamma = 0 \quad .$$

This is the orthogonality relation which states that $x^i_{\alpha,\beta}$ 94 is a space vector normal to the surface, and hence it is directed along the unit normal n^i. Consequently, there exists a set of functions $b_{\alpha\beta}$ such that

$$x^i_{\alpha,\beta} = b_{\alpha\beta}\, n^i \quad .$$

The quantities $b_{\alpha\beta}$ are the components of a symmetric surface tensor, and the differential quadratic form

$$B \equiv b_{\alpha\beta}\, du^\alpha\, du^\beta$$

is the desired second fundamental form.

Now since

$n_j n^i = \delta^i_j$ 97, and

$n_i = \hat{g}_{ij} n^j$ 98, then

$$b_{\alpha\beta} = \hat{g}_{ij}\, x^i_{\alpha,\beta}\, n^j \quad ;$$

but since

$n_i = \frac{1}{2} \varepsilon^{\alpha\beta}\, \varepsilon_{ijk}\, x^j_\alpha\, x^k_\beta$ 100, then

$$b_{\alpha\beta} = \frac{1}{2} \varepsilon^{\gamma\delta}\, \varepsilon_{ijk}\, x^i_{\alpha,\beta}\, x^j_\gamma\, x^k_\delta \quad . \tag{7.38}$$

We now have, in Eqn.s (7.8) and (7.38), the formulas necessary to determine the first and second fundamental quadratic forms for our system constrained to a four-dimensional hypersurface. Our objective is to show that by appropriately constraining our system we arrive at the Navier-Stokes equations. Let us determine the first fundamental quadratic form.

First recall that our system was restricted so that $x^4 = x^4(x^o, x^1, x^2, x^3)$ or the mass density is a function of space and time; then we have the relations

$$x^0 = u^0,$$
$$x^1 = u^1,$$
$$x^2 = u^2,$$
$$x^3 = u^3,$$

and

$$x^4 = f(x^0, x^1, x^2, x^3) = f(u^0, u^1, u^2, u^3) \quad .$$

Since Eqn. (7.8) is

$$A_{\alpha\beta} = \hat{g}_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u_\beta} = \hat{g}_{ij}\, x^i_\alpha\, x^j_\beta \quad ,$$

then

$$A_{00} = g_{00} = 2\,g_{04}\,f_0 + g_{44}(f_0)^2 \quad ,$$

where

$$f_0 \equiv \frac{\partial f}{\partial u^0}$$

In a similar fashion we may determine the remaining coefficients and find that

$$A_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta} \tag{7.39}$$

where

$$h_{\alpha\beta} = 2\,\hat{g}_{\alpha 4}\,f_\beta + \hat{g}_{44}\,f_\alpha\,f_\beta \quad ; \quad \alpha, \beta = 0, 1, 2, 3 \quad , \tag{7.40}$$

where the $h_{\alpha\beta}$ are functions of the partial derivatives of the mass density with respect to space and time in addition to space and time from the $\hat{g}_{i4}$ 109 where i = 0, 1, 2, 3, 4.

Though we may use Eqn. (7.38) to determine the metric coefficients for the second fundamental quadratic form, it is not necessary for the current presentation.

The hypersurface which is embedded in the five-dimensional space is a four-dimensional curvilinear space-time manifold. Thus the relativistic hydrodynamic equations are applicable here so long as the metric coefficients are determined as coefficients of the hypersurface quadratic form.

The complete energy-momentum tensor for a fluid in a flat Riemannian space-time manifold is given by

$$T^{\alpha\beta} = \gamma\,\dot{u}^\alpha\,\dot{u}^\beta + \frac{P}{c^2}(\dot{u}^\alpha\,\dot{u}^\beta - g^{\alpha\beta}) \tag{7.41}$$

where

$\dot{u}^\alpha \equiv \dfrac{du^\alpha}{ds}$ 111, s is the arc length. Then based upon this energy momentum tensor the flow of a fluid under the effect of its own internal pressure force is given by setting the divergence of Eqn. (7.41) equal to zero, or

$$T^{\alpha\beta}{}_{,\beta} = 0 \quad .$$

(7.42)

If we reduce Eqn. (7.41) to the non-relativistic limit, the use of Eqn. (7.42) gives us

$$g^{\beta\delta}\tau_{\alpha\beta,\delta} = \gamma^{a}_{\alpha} \quad , \quad \alpha, \beta, \delta = 1, 2, 3 \quad ,$$

where $\tau^{\alpha\beta} = Pg^{\alpha\beta}$ is the three-dimensional stress tensor of an ideal fluid.

If in Eqn. (7.41) we use the fact that the metric coefficients for the hypersurface may be written as the sum of Eqn. (7.40), then we have

$$T^{\alpha\beta} = \gamma\, \dot{u}^{\alpha}\dot{u}^{\beta} + \frac{P}{c^2}\left(\dot{u}^{\alpha}\dot{U}^{\beta} - \hat{g}^{\alpha\beta} - h^{\alpha\beta}\right) \quad ,$$

(7.43)

where it must be remembered that the

$\dot{u}^{\alpha}\dot{u}^{\beta}$ 115 are also dependent upon this same sum. In the non-relativistic limit the effects of this sum of metric tensors appear as a sum in the stress tensor

$$\tau^{\alpha\beta} = -P\,\hat{g}^{\alpha\beta} - Ph^{\alpha\beta} \quad , \ \alpha, \beta = 1, 2, 3 \quad .$$

(7.44)

Recall that the

$\hat{g}^{\alpha\beta}$ 117 refer to the three-dimensional space viewed from the five-dimensional manifold. The $h^{\alpha\beta}$, however, contain the information about the surface embedded in the five-dimensional space. If we then associate the tensor

$$t^{\alpha\beta} \equiv -Ph^{\alpha\beta}$$

(7.45)

with the viscous stresses, we are saying that the viscous stresses depend upon the geometric character of the hypersurface.

In the limit of small displacements we write the strain velocity tensor as

$$\dot{e}_{\alpha\beta} = \frac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha}\right) \quad .$$

Then the first order coefficients of viscosity are related to the strain velocity tensor and viscous stresses according to

$$t^{\alpha\beta} = \frac{c^{\alpha\beta\delta n}}{2}\left(v_{\delta,n} + v_{n,\delta}\right) \ .$$

(7.46)

If we then use Eqn. (7.45) in Eqn. (7.46), we find that the relationship between the geometric character of the hypersurface and the viscous coefficients is given by

$$- Ph^{\alpha\beta} = \frac{c^{\alpha\beta\delta n}}{2}\left(v_{\delta,n} + v_{n,\delta}\right) \ .$$

(7.47)

Equation (7.47) then expresses the functional dependence of the viscous coefficients upon the strain velocities, pressure, mass density, and their derivatives.

7.4. Relativistic Hydrodynamics.

By viewing classical hydrodynamics to be given by the embedding of a four-dimensional hypersurface within a five-dimensional manifold, the association Eqn. (7.47) between the geometric properties of the hypersurface and the viscous coefficients could be tentatively made. We may now go back and develop this relationship more completely.

The hypersurface, which becomes embedded in the five-dimensional manifold by the restriction that $x^4 = x^4(x^0, x^1, x^2, x^3)$ is a four-dimensional relativistic manifold. Thus, for the surface we may use the relativistic energy-momentum tensor, which is

$$T^{\mu\nu} = \gamma \ u^{\mu\nu} u + \frac{P}{c^2}\left(u^{\mu} u^{\nu} - g^{\mu\nu}\right) \ ,$$

(7.48)

where

$u^{\mu} \equiv \frac{dx^{\mu}}{dx^{0}}$ 123 and $\mu\nu$ = 0, 1, 2, 3. The divergence of Eqn. (7.48) yields the flow equations for a fluid under the effects of its own internal pressure.

However, from the viewpoint of the Dynamic Theory, the surface metric coefficients may be written in terms of the metric coefficients of the first four space coordinates as given by Eqn.s (7.40) and (7.41), or

$$A_{\alpha\beta} = \hat{g}_{\alpha\beta} \ , \ \alpha, \beta = 0, 1, 2, 3 \ .$$

Thus, the square of the arc length for the entropy manifold may be written as

$$\left(dq^0\right)^2 = A_{\alpha\beta}\, dx^\alpha\, dx^\beta = \hat{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta + h_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

or, if

$u^\alpha \equiv \dfrac{dx^\alpha}{dq^0}$ 126, then

$$1 = A_{\alpha\beta}\, u^\alpha\, u^\beta = \hat{g}_{\alpha\beta}\, u^\alpha\, u^\beta + h_{\alpha\beta}\, u^\alpha\, U^\beta \quad .$$

Then on the hypersurface the energy momentum tensor would become

$$T^{\alpha\beta} = \gamma\; u^\alpha\, u^\beta + \frac{P}{c^2}\left(u^\alpha\, u^\beta - A^{\alpha\beta}\right)$$

or

$$T^{\alpha\beta} = \gamma\; u^\alpha\, u^\beta + \frac{P}{c^2}\left(u^\alpha\, u^\beta - \hat{g}_{\alpha\beta} - h^{\alpha\beta}\right) \quad .$$

(7.49)

Since the surface coordinates, x^α, are the same as the first four coordinates of the surrounding space, the velocities u^α are the same whether considered as surface or space vectors. The difference between the surface view and a four-dimensional space view appears in the metric coefficients. Thus, while the square of the arc element on the surface is unity, the square of the arc element in the surrounding space is not, or

$$1 = A_{\alpha\beta}\, u^\alpha\, u^\beta$$

but

$$\hat{g}_{\alpha\beta}\, u^\alpha\, u^\beta = 1 - h_{\alpha\beta}\, u^\alpha\, u^\beta \quad .$$

7.5. Classical Hydrodynamics.

Suppose we consider the metric given by
$\hat{g}_{\alpha\beta}$ 132 to be a flat space then because of Eqn. (7.49) we may write

$$T^{\alpha\beta} = \gamma\; u^\alpha\, u^\beta + \frac{P}{c^2}\left(u^\alpha\, u^\beta - \hat{g}^{\alpha\beta}\right) - \frac{P}{c^2} h^{\alpha\beta} \quad .$$

If we then form the space divergence

$$T^{0\nu},_\nu = \frac{1}{c}\frac{\partial}{\partial t}\left\{\gamma - \frac{P}{c^2}h^{00}\right\} + \frac{\partial}{\partial x^\alpha}\left\{\gamma\frac{u^\alpha}{c} + \frac{P}{c^2}\left(\frac{u^\alpha}{c} - H^{0\alpha}\right)\right\} = 0 \quad ,$$

this may be written as

$$\frac{1}{c}\left[\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v})\right] - \frac{1}{c^2}\frac{\partial(Ph^{00})}{\partial t} + \frac{1}{c^2}\bullet(P\bar{v}) - \frac{1}{c^2}\bar{\Delta}\bullet(Ph^{\cdot 0}) = 0 \quad ,$$

where h^0 has components $h^{0\alpha}$, = 1, 2, 3. Therefore,

$$\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v}) = \frac{1}{c}\bar{\Delta}\bullet(P\bar{v}) + \frac{1}{c^2}\left[\frac{\partial(Ph^{00}}{\partial t} + \bar{\Delta}\bullet(\overline{Ph_0})\right] \quad ,$$

so that if h^ν is a four-vector with components $h^{0\nu} \equiv h^\nu$ 137,
then

$$\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v}) = -\frac{1}{c}\bar{\Delta}P\bullet\bar{v} - \frac{P}{c}v^\alpha,_\alpha + \frac{1}{c^2}(Ph^\nu),_\nu \quad .$$

The remaining components of the divergence are given by

$$T^{\alpha\nu},_\nu = \frac{1}{c}\frac{\partial}{\partial t}\left\{\gamma\frac{u^\alpha}{c} + \frac{Pv^\alpha}{c^3} - \frac{P}{c^2}h^{\alpha 0}\right\}$$
$$+ \frac{\partial}{\partial x^\beta}\left\{\gamma\frac{v^\alpha v^\beta}{c^2} + \frac{P}{c^2}\left(\frac{v^\alpha v^\beta}{c^2} + \delta^{\alpha\beta}\right) - \frac{Ph^{\alpha\beta}}{c^2} = 0 \quad ,\right.$$

which may be rearranged to read

$$\gamma\left[\frac{\partial v^\alpha}{\partial t} + \bar{v}\bullet\Delta\, v^\alpha\right] = -\frac{\partial P}{\partial x^\alpha} - v^\alpha\left[\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v})\right] + \frac{1}{c}\frac{\partial(Ph^{\alpha 0})}{\partial t} + \frac{\partial(Ph^{\alpha\beta})}{\partial x^\beta}$$
$$-\frac{1}{c^2}\left[\frac{\partial(Pv^\alpha)}{\partial t} + \bar{\Delta}\bullet(P\,v^\alpha\,\bar{v})\right] \quad .$$

If we look at the non-relativistic limit, then, by neglecting the terms P(v/c), we get

$$\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v}) = \frac{1}{c^2}\left[\frac{1}{c}\frac{\partial(Ph^{00})}{\partial t} + \bar{\Delta}\bullet(Ph^{\cdot 0})\right] \quad .$$

The multiplicative factor $1/c^2$ on the right-hand side suggests that

$$\frac{\partial\gamma}{\partial t} + \bar{\Delta}\bullet(\gamma\,\bar{v}) \cong 0 \quad ,$$

which is the assumption we chose to place our system on a particular surface. This corresponds to a classical system where conservation of mass is assumed. Therefore, on the surface of a curve specified by

$$T^{\mu\nu}{}_{,\nu} = 0 \quad ,$$

we must then have

$$\gamma\left[\frac{\partial v^{\alpha}}{\partial t} + \bar{v} \bullet \bar{\Delta}\, v^{\alpha}\right] = -\frac{\partial P}{\partial x^{\alpha}} + \frac{1}{c}\frac{\partial (Ph^{\alpha 0})}{\partial t} + \frac{\partial (Ph^{\alpha\beta})}{\partial x^{\beta}}$$

or

$$\gamma\ a^{\alpha} = -\frac{\partial P}{\partial x^{\alpha}} + \frac{1}{c}\frac{\partial (Ph^{\alpha 0})}{\partial t} + \frac{\partial (Ph^{\alpha\beta})}{\partial x^{\beta}}$$
$$= -\frac{\partial P}{\partial x^{\alpha}} + \frac{1}{c}\frac{\partial (Ph^{\alpha 0})}{\partial t} + (Ph^{\alpha\beta})_{,\beta} \quad , \quad \beta = 1, 2, 3 \quad .$$

Thus, we may write

$$\gamma\ a^{\alpha} = \tau^{\alpha\beta}{}_{,\beta}$$

where

$$\tau^{\alpha\beta} = -P\,\hat{g}^{\alpha\beta} + Ph^{\alpha\beta} = -P(\hat{g}^{\alpha\beta} - h^{\alpha\beta}) \quad . \tag{7.50}$$

The term
$\frac{1}{c}\frac{\partial (Ph^{\alpha 0})}{\partial t}$ 148 has been neglected in Eqn. (7.50).

Thus we see that the geometric character of the hypersurface, contained in the term $Ph^{\alpha\beta}$, behaves as if it were a viscous effect to be added to the normal viscous effects. Recalling Eqn. (7.41), it may be seen that the viscous-like effects of the geometry of the hypersurface depend upon the density gradient. If these terms exist, they must be very small in everyday phenomena. Yet if we consider phenomenon which involve very large density gradients, these terms could become large enough to see.

7.6. Shock Waves.

One field of physical phenomena that displays large density gradients is shock waves. Therefore let us take a quick look at the effect of these additional terms on the description of a shock front for a steady, one-dimensional shock.

The total stress in a steady, one-dimensional shock would be given by

$$\sigma = P\left\{1 - \left(\frac{1}{a_o^2}\right)\left(\frac{\partial \gamma}{\partial x}\right)^2\right\} + \eta\left(\frac{du}{dx}\right) \quad ,$$

when g_{11} = 1 and h_{11} is evaluated using Eqn. (7.41). However, for a steady shock we also have the jump conditions

$$\gamma u = k_1 \ ,$$
$$k_1 u + \sigma = k_2 \ ,$$

and

$$k_1^2 E - \frac{\sigma^2}{2} = k_3 \ .$$

These equations represent the conservation of mass, momentum, and energy. By using the conservation of mass relation we may write the total stress as

$$\sigma = P + \eta_{eff}\left(\frac{du}{dx}\right) \ ,$$

where

$$\eta_{eff} \equiv \eta - \frac{P k_1^2}{a_0^2 u^4}\left(\frac{du}{dx}\right) \tag{7.51}$$

may be called the effective viscous coefficient. Since within the shock front the velocity gradient du/dx is negative, we see that the effective viscous coefficient acts so as to thicken the shock front when compared to the classical viscous coefficient .

Using the second jump condition, an expression for the velocity gradient is

$$\frac{du}{dx} = \frac{a_0^2 u^4 \eta}{2 P k_1^2}\left\{1 - \sqrt{1 + \frac{4 P k_1^2}{a_0^2 u^4 \eta^2}[P - k_2 + k_1 u]}\right\}, \tag{7.52}$$

which may be approximated by

$$\frac{du}{dx} = -\left(\frac{1}{\eta}\right)[P - k_2 + k_1 u]\left\{1 - \frac{P k_1^2}{a_0^2 u^4 \eta^2}[P - k_2 + k_1 u]\right\}. \tag{7.53}$$

The effect of the correction term on the velocity gradient is seen in Eqn. (7.53), because the multiplicative factor outside the brackets is the classical expression for the negative velocity gradient. The effect of the correction term lessens the negative velocity gradient and extends the shock front.

The effect of the correction term in Eqn. (7.51) is estimated by considering the strong shock dependence of pressure upon shock velocities. For instance, the shock pressure, from the jump conditions, is

$$P = \gamma_o U u_p \ . \tag{7.54}$$

If the shock velocity is related linearly to the particle velocity as the assumed solid equation of state, $U = c_o + su_p$, then Eqn. (7.54) becomes

$$P = \frac{\gamma_0 U}{s}(U - c_o) .$$

Thus, for strong shocks, P varies approximately as the square of the shock velocity.

Consider Eqn. (7.52) or (7.53). From either of these equations, the velocity gradient varies as the square of the shock velcoity. Using these two conclusions in Eqn. (7.51) for the total viscosity η_{eff} and remembering that the integration contant k_1 is given by $-\gamma_o U$, the effective viscosity varies approximately as the square of the shock velocity or, essently , as the pressure.

The conclusion is that if the effective viscosity varies with the pressure, an increase by the same factor of 10^3 must be accompanied by a viscosity increase by the same factor of 10^3. This explains the apparent discrepancy between the low and high pressure aluminum viscous effects. For instance, the Asay-Bertholf limits are:

$$P = 25 \text{ GPa} \qquad \eta > 40 \text{ poise}$$
$$P = 36 \text{ GPa} \qquad \eta < 2{,}500 \text{ poise} .$$

Another experiment places an upper limit of 10^3 poise for a shock pressure of 40 GPa. If 10^2 poise is considered representative of the viscosity when P □ 10 GPa, then from Eqn. (7.51), a pressure of 10^3-10^4 GPa must be accompanied by a viscous effect of 10^4-10^5 poise.

This total viscosity estimate is supported by numerical integration across the shock front using the Tillotson equation of state for aluminum. The classically predicted rise times for shocks of 40 GPa with η = 575 p and $5\text{x}10^3$ GPa for $\eta = 5\text{x}10^4$p are duplicated by using the effective viscosity experssion in Eqn. (7.51) with η = 1.0 p and a_0 . 365 g/cm^4.

Thus, the Dynamic Theory correlates these data points that appear contradictory by classical theory. Further, these data points provide an estimate of the new universal constant appearing in the Dynamic Theory.

This value of a_0 provides an estimate of other predictions of the theory in fields other than shock waves.

7.7. Mass Conservative Electrodynamics

One of the incentives for seeking to determine whether the five equations of motion were generalizations of the classical hydrodynamic equations was the possibility of shedding new light upon fusion plasmas. Now before mass conversion is accomplished the plasma must reach certain conditions. The attainment of these conditions involve electromagnetic fields not encountered in usual circumstances on earth. If the Dynamic Theory is to be believed, then perhaps it may provide new insight into the attainment of the appropriate conditions before mass conversion begins.

The following development still assumes conservation of mass in order to see the geometry of the hypersurface for a system under the influence of electromagnetic fields.

Suppose we now describe the behavior of charged matter under the influence of an electromagnetic field from the viewpoint of the Dynamic Theory. From this viewpoint the conservation of mass has the effect of restricting our system to a four-dimensional hypersurface which is embedded in the five-dimensional manifold of space, time, and mass density.

Since we desire to consider the effects of an electromagnetic field we must consider a gauge function. When a gauge function exists, the square of the arc length in the entropy space is related to the square of the arc length in the sigma space by

$$\left(dq^0\right)^2 = \hat{g}_{ij}\, dx^i\, dx^j = \left(\frac{1}{h_{00}}\right)\hat{g}_{ij}\, dx^i\, dx^j = \left(\frac{1}{h_{00}}\right)(d\sigma)^2 \quad .$$

When the system is restricted to a hypersurface by the relation $x^4 = x^4(x^0, x^1, x^2, x^3)$, then the entropy surface may be written as

$$\left(dq^0\right)^2 = \hat{a}_{\alpha\beta}\, du^\alpha\, du^\beta \quad ,$$

where

$$\hat{a}_{\alpha\beta} = \hat{g}_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x_i}{\partial u^\beta} = g_{ij}\, x^i_\alpha\, x^j_\beta \quad .$$

Likewise for the sigma surface

$$(d\sigma)^2 = \hat{a}_{\alpha\beta}\, du^\alpha\, du^\beta$$

where

$$\hat{a}_{\alpha\beta} = \hat{g}_{ij}\, x^i_\alpha\, x^j_\beta \quad .$$

Thus, we have

$$\hat{a}_{\alpha\beta} = \left(\frac{1}{h_{00}}\right)\hat{a}_{\alpha\beta} \quad .$$

The principle of increasing entropy requires that the equations of motion be geodesics in the entropy space but they will appear as equations involving forces in the sigma space. We desire to expose these forces and therefore should work in the sigma space. Our objective then is to determine the effect of embedding a four-dimensional surface given by $x^4 = x^4(x^0, x^1, x^2, x^3)$ in the sigma space and thus obtain a sigma surface describing a system subjected to the classical conservation of mass restriction.

Having previously determined the metric coefficients for the entropy space by Eqns. (7.39) and (7.40) we may write the coefficients for the sigma surface as

$$\hat{a}_{\alpha\beta} = h_{00}\, \hat{a}_{\alpha\beta} = h_{00}\left\lfloor \hat{g}_{\alpha\beta} + h_{\alpha\beta} \right\rfloor \; .$$

However by considering the effects of the electromagnetic field as a force we must first consider the space field tensor:

$$F_{ij} = \begin{vmatrix} 0\; E_1\; E_2\; E_3\; V_0 \\ -E_1\; 0\; B_3\; -B_2\; V_1 \\ -E_2\; -B_3\; 0\; B_1\; V_2 \\ -E_3\; B_2\; -B_1\; 0\; V_3 \\ -V_0\; -V_1\; -V_2\; -V_3\; 0 \end{vmatrix}$$

If we restrict ourselves to the classical field quantities E and B and for the moment assume that the field quantities V_4 and V are zero or negligible, then we obtain only the effects of the hypersurface viewpoint. This assumption seems reasonable considering the interpretation of the new field quantities are gravitational effects. Under this assumption our field tensor becomes

$$F_{ij} = \begin{vmatrix} 0\; E_1\; E_2\; E_3\; 0 \\ -E_1\; 0\; B_3\; -B_2\; 0 \\ -E_2\; -B_3\; 0\; B_1\; 0 \\ -E_3\; B_2\; -B_1\; 0\; 0 \\ 0\; 0\; 0\; 0\; 0 \end{vmatrix}$$

We can now use this space field tensor to determine the appearance of the fields when viewed from the surface. The surface field tensor will be given by

$$F_{\alpha\beta} = F_{ij}\, x_\alpha^i\, x_\beta^j \quad .$$

But since $x_\alpha^i = \delta_{\alpha i}$ for i, = 0, 1, 2, 3 and $x_\alpha^4 = f_\alpha$, the surface field tensor of a purely electromagnetic space field tensor is only the four-dimensional portion of the space field tensor since $F_{i4} = 0$ for i = 0, 1, 2, 3, 4.

Thus when we use the relativistic energy-momentum tensor for the surface, we have

$$T^{\mu\nu} = \gamma\, u^\mu u^\nu + \frac{1}{c^2}\left[F^\mu_\alpha\, F^{\alpha\nu} + \frac{1}{4}\hat{a}^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta} \right] \ , \tag{7.55}$$

which is the relativistic energy-momentum tensor for matter under the influence of electromagnetic fields. But since
$\hat{a}_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$ 169, then Eqn. (7.55) becomes

$$T^{\mu\nu} = \gamma\, u^\mu u^\nu + \frac{1}{c^2}\left[F^\mu{}_\alpha\, F^{\alpha\nu} + \frac{1}{4}\left(\hat{g}^{\mu\nu} + h^{\mu\nu}\right) F^{\alpha\beta}\, F_{\alpha\beta} \right]$$

or

$$T^{\mu\nu} = T^{\mu\nu}_{rel} + T^{\mu\nu}_{geo} \tag{7.56}$$

where

$$T^{\mu\nu}_{rel} \equiv \gamma\, u^\mu u^\nu + \frac{1}{c^2}\left[F^\mu{}_\alpha\, F^{\alpha\nu} + \frac{1}{4}\hat{g}^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta} \right]$$

is the four-dimensional space relativistic energy momentum tensor and

$$T^{\mu\nu}_{geo} \equiv \left(\frac{1}{4c^2} \right) h^{\mu\nu}\, F^{\alpha\beta}\, F_{\alpha\beta}$$

is the portion of the energy-momentum tensor which contains the geometrical properties of the hypersurface.

From Eqn. (7.56) we can say that the Dynamic Theory has the appearance of adding a term to the relativistic energy-momentum tensor. This term contains the geometrical character of the surface and represents the difference between the appearance of the energy-momentum tensor when viewed from the surrounding space as compared to the view from the hypersurface.

If we take the divergence of the energy-momentum tensor Eqn. (7.56), we have

$$T^{\mu\nu}{}_{,\nu} = T^{\mu\nu}_{rel\ ,\nu} + T^{\mu\nu}_{geo\ ,\nu} \ .$$

The additional force terms from the surface geometry are given by

$$\left(\frac{1}{4\,c^2}\right)\left(h^{\mu\nu}\,F^{\alpha\beta}\,F_{\alpha\beta}\right),\nu = F^{\mu} \ .$$

But if we define

$$F^{\alpha\beta}\,F_{\alpha\beta} \equiv -16\tau\xi$$

(7.57)

as the electromagnetic energy density, where

$$\xi = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

then the geometric energy-momentum tensor becomes

$$T^{\mu\nu}_{geo} = \frac{-4\pi\ h^{\mu\nu}\,\xi}{c^2}$$

and the additional forces are given by

$$F^{\mu} = -\frac{4\pi}{c^2}\left(h^{\mu\nu}\,\xi\right),\nu$$

We may also look at the radiation pressure predicted by the Dynamic Theory to see how the surface restriction affects the relativistic prediction of radiation pressure.

The relativistic radiation pressure is taken as one third of the three-dimensional Maxwell stress tensor which is the space portion of the energy-momentum tensor, or

$$T^{M}_{\alpha\beta} = \frac{1}{4\pi}\left(E_\alpha\,E_\beta + B_\alpha\,B_\beta\right) - \delta_{\alpha\beta}\,\xi$$

where α, β = 1, 2, 3.

To get the equivalent stress tensor for the Dynamic radiation pressure we must add the space portion of Eqn. (7.57) so that the total stress tensor becomes

$$\begin{aligned} T_{\alpha\beta} &= \frac{1}{4\pi}\left(E_\alpha\,E_\beta + B_\alpha\,B_\beta\right) - \delta_{\alpha\beta}\,\xi - 4\,h_{\alpha\beta}\,\xi \\ &= \frac{1}{4\pi}\left(E_\alpha\,E_\beta + B_\alpha\,B_\beta\right) - \xi\left(\delta_{\alpha\beta} + h_{\alpha\beta}\right) \ . \end{aligned}$$

We can then obtain the negative of the trace by

$$-\{T\} = -\left[\frac{1}{4\pi}\left(E^2+B^2\right) - 3\xi - \left(h_{11}+h_{22}+h_{33}\right)\xi\right]$$
$$= -\left[-\xi - \left(h_{11}+h_{22}+h_{33}\right)\xi\right] \ .$$

The radiation pressure is then given by

$$P = \frac{\xi}{3}[1+h_{11}+h_{22}+h_{33}] \ . \tag{7.58}$$

The first term in Eqn. (7.58) is the classical radiation pressure in electrodynamics. The remaining three terms give the difference between the pressure predicted by the Dynamic Theory and the classical prediction. To determine what this difference is let us restrict our system to again be very near equilibrium so that the $g_{\alpha 4}$ = 0 for α = 0, 1, 2, 3 and g_{44} = constant. Thus we have a flat space. For this space the

$$h_{\alpha\alpha} = \frac{\hat{g}_{44}}{a_0^2}\left(\frac{\partial\gamma}{\partial x^\alpha}\right)^2$$

from Eqn. (7.58) and g_{44} = -1. Thus

$$h_{11}+h_{22}+h_{33} = -\frac{1}{a_0^2}\left[\left(\frac{\partial\gamma}{\partial x^1}\right)^2 + \left(\frac{\partial\gamma}{\partial x^3}\right)^2 + \left(\frac{\partial\gamma}{\partial x^3}\right)^2\right] \ . \tag{7.59}$$

By substituting Eqn. (7.59) into Eqn. (7.58) the pressure becomes

$$P = \frac{\xi}{3}\left\{1 - \frac{1}{a_0^2}\left[\left(\frac{\partial\gamma}{\partial x^1}\right)^2 + \left(\frac{\partial\gamma}{\partial x^2}\right) + \left(\frac{\partial\gamma}{\partial x^3}\right)^2\right]\right\} \ .$$

However, since the classical pressure is given by
$P_c = \frac{\xi}{3}$ 187, then the pressure predicted by the Dynamic Theory becomes

$$P_D = P_c\left\{1 - \left(\frac{1}{a_0^2}\right)\left[\left(\frac{\partial\gamma}{\partial x^1}\right)^2 + \left(\frac{\partial\gamma}{\partial x^2}\right)^2 + \left(\frac{\partial\gamma}{\partial x^3}\right)^2\right]\right\} \ .$$

We see then that the Dynamic Theory predicts a decrease in the radiation pressure as a result of viewing the system to be restricted to a four-dimensional hypersurface embedded in a five-dimensional space. The amount of this decrease in pressure depends upon the gradient of the mass density and the constant a_0. Once the constant a_0 is determined, then the deviation in predicted pressures can be specified.

This prediction should appear in attempts to use electrodynamic forces to control ionized plasmas and perhaps there are large enough density gradients for these predictions to show up in cosmological events.

References:

*A. J. McConnell, Absolute Differential Calculus, London, 1931, Chapters IV - XVI.