Lamb Shift of a Non-Singular Electrostatic Potential

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The non-singular potential may be solved in the non-relativistic Quantum mechanics as shown in the article, Williams, P.E., "Mechanical Entropy and its Implications," *Entropy* **2001**, *3*, 76-115. http://www.mdpi.org/entropy/list01.htm#new

Some of the text of that article which shows the solution of the radial equation and, therefore, the energy levels is repeated below:

The equation for ψ_r becomes

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\psi_r}{dr}\right]+\frac{2\mu}{\hbar^2}\left[E-E_c-V(r)-V_c(r)\right]\psi_r=\frac{\ell(\ell+1)\psi_r}{r^2}.$$

Recognizing the potential difficulties of obtaining solutions with both $V(r)$ and $V_c(r)$ having transcendental terms, an investigation of the influence of the non-singular potentials may be conducted by assuming $\lambda_1 << r$ and $\lambda_2 << r$.

Following the method of solving Schrödinger's time-independent equation, the following definitions:

$$\rho=2\beta r;\ \beta^2=\frac{2\mu(E-E_c)}{\hbar^2};\ \gamma=\frac{\mu Ze^2}{4\pi\varepsilon_0\hbar^2\beta}$$

and

$$\lambda=\frac{2\mu Ze^2}{4\pi\varepsilon_0\hbar^2}\left[\frac{(2m_1+m_2)\lambda_1}{(m_1+m_2)}-\frac{m_1\lambda_2}{(m_1+m_2)}\right],$$

together with the trial solution,

$$\psi_r(\rho)=e^{-\rho/2}F(\rho)=e^{-\rho/2}\left[\rho_s\sum_{k=0}^{\infty}a_k\rho^k\right],\quad a_0\neq 0 \text{ and } s\geq 0,$$

requires that two relations must be met; namely

$$s(s+1)=\ell(\ell+1)+\lambda$$

and

$$a_{j+1}=\frac{s+j+1-\gamma}{(s+j+1)(s+j+2)-[\ell(\ell+1)+\lambda]}a_j.$$

In order to satisfy $s \geq 0$, $s = l+\delta$ where

$$\delta=-\left(\ell+\frac{1}{2}\right)\left[1-\sqrt{1+\frac{\lambda}{\left(\ell+\frac{1}{2}\right)^2}}\right].$$

Further, the series will terminate if γ-δ=n is an integer. The solution may then be put into the form

$$E - E_c = \frac{E_n}{\left(1 + \frac{2\delta}{n}\right)}; \; where \; E_n = \frac{-\mu Z^2 e^4}{(4\pi\varepsilon_0)^2 2\hbar^2 n^2}.$$

The energy levels found here depend both upon the principle quantum number, *n*, and the quantum number *l*. This means that the energy levels for a specific *n* will be split into separate energy levels for different values of *l*. This is precisely the split that was given the name Lamb Shift after it was detected. We would now like to estimate the Lamb Shift that the non-singular potential would predict.

Also in the article it was shown that the lambdas were approximately proportional to the mass so that $\lambda_1 \approx m_1 \lambda_2 / m_2$. Given this value for λ_2 the expression for λ may be written as

$$\lambda = \frac{4 m_1^3 Z e^2}{4\pi\varepsilon_0 \hbar^2 (m_1 + m_2)^2} \lambda_2,$$

Now we wish to look at the energy difference between *n=2; l=0* and *n=2; l=1*. This means that, when $\lambda << 1$ we may approximate the Lamb Shift energy as

$$\Delta E_{LS} = \left(E - E_c\right)_{l=0} - \left(E - E_c\right)_{l=1} = \left\{ \frac{1}{(1+\lambda)} - \frac{1}{\left(1 + \frac{\lambda}{3}\right)} \right\} E_2$$

$$= \left\{ \frac{-2\lambda}{(1+\lambda)(3+\lambda)} \right\} E_2$$

$$\approx \left(\frac{-2\lambda}{3} \right) E_2$$

Now we may substitute our expression for λ into the Lamb shift energy expression to get

$$\Delta E_{LS} \approx \left(\frac{-2}{3} \frac{4 m_1^3 Z e^2}{4\pi\varepsilon_0 \hbar^2 (m_1 + m_2)^2} \right) E_2 \lambda_2$$

Thus, it may be seen that a measurement of the Lamb Shift energy is proportional to λ_2 the more massive of the two particles. When $m_1 << m_2$ the Lamb Shift energy may be written as

$$\Delta E_{LS} \approx \left[-\frac{8}{3} \frac{m_1^3 Z e^2}{\left(4\pi\varepsilon_0\right) \hbar^2 m_2^2} \lambda_2 \right] E_2 .$$

In this expression for the Lamb Shift the energy difference is only due to the non-singular potential and does not include any energy difference due to fine structure splitting or hyperfine and radiative corrections. Perhaps the most interesting feature is that this energy difference varies as m_1 to the fourth power. This means that such a predicted energy difference increases 8 orders of magnitude when the mass of m_1 increases by 2 orders of magnitude. This is an indication of how strongly the use of muons amplifies the Lamb Shift energy.

An investigation of the real difference the non-singular potential should predict for the Lamb Shift must await the determination of the solutions to the relativistic equation for the non-singular potential which is given by the much more complicated equation, also given in the above article as

$$
\begin{aligned}
&[(i\partial_{1\mu}-e_1A_{2\mu}+i\partial_{2\mu}-e_2A_{1\mu})(i\partial_1^{\mu}+e_1A_2^{\mu}+i\partial_2^{\mu}-e_2A_1^{\mu})-(m_1+m_2)^2\\
&\quad -ie_1\sigma^{\mu\nu}(\partial_{1\mu}A_{2\nu}-\partial_{1\nu}A_{2\mu})-ie_1\sigma^{\mu\nu}(\partial_{2\mu}A_{2\nu}-\partial_{2\nu}A_{2\mu}) \qquad (1)\\
&\quad -e_2\sigma^{\mu\nu}(\partial_{1\mu}A_{1\nu}-\partial_{1\nu}A_{1\mu})-ie_2\sigma^{\mu\nu}(\partial_{2\mu}A_{1\nu}-\partial_{2\nu}A_{1\mu})]\psi=0.
\end{aligned}
$$

From these solutions the fine structure and hyperfine corrections due to the non-singular potential may be found.

As for radiative terms the difference in predicted energy between the coulomb and the non-singular potential may be found by using the non-singular potential in the QED calculations.

Electronic versus Muonic Lamb Shift

The point to be displayed here is the difference that the larger mass of the muon makes in the prediction of the Lamb Shift for muonic hydrogen compared to electronic hydrogen. Starting by putting in the values for the muonic hydrogen we find

$$
\begin{aligned}
\Delta E_{LS\mu} &\approx \left[-\frac{8}{3}\frac{\left(8.99\times10^{9}\,\text{nt-m}^2/\text{coul}^2\right)\left(1.883\times10^{-28}\,\text{kg}\right)^3\left(1.6\times10^{-19}\,\text{coul}\right)^2}{\left(1.05\times10^{-34}\,\text{joule-sec}\right)^2\left(1.67\times10^{-27}\,\text{kg}\right)^2}\lambda_2\right]\left(-\frac{13.58\text{eV}}{4}\right)\left(\frac{105.66\text{MeV}}{0.511\text{MeV}}\right)\\
&=\left[-\frac{8}{3}\frac{\left(8.99\times10^{9}\,\text{nt-m}^2/\text{coul}^2\right)\left(6.677\times10^{-84}\,\text{kg}^3\right)\left(2.56\times10^{-38}\,\text{coul}^2\right)}{\left(1.103\times10^{-68}\,\text{joule}^2\text{-sec}^2\right)\left(2.789\times10^{-54}\,\text{kg}^2\right)}\right]\left(-701.99\text{eV}\right)\lambda_2\\
&=\left[-1.33\times10^{11}\,\frac{1}{\text{m}}\right]\left(-701.99\text{eV}\right)\lambda_2\\
&=\left(9.35675\times10^{13}\,\frac{\text{eV}}{\text{m}}\right)\lambda_2
\end{aligned}
$$

Since the mass of the electron is some 200 times smaller than that of the muon, the electronic hydrogen would have a non-singular potential Lamb Shift of

$$
\Delta E_{LSe}\approx\left(\frac{0.511\text{MeV}}{105.66\text{MeV}}\right)^4\left(9.35675\times10^{13}\,\frac{\text{eV}}{\text{m}}\right)\lambda_2=\left(5.1188\times10^{4}\,\frac{\text{eV}}{\text{m}}\right)\lambda_2
$$

If we now introduce the value of λ_2 determined by the deuterium nucleus mass, we find

$$
\Delta E_{LS}\approx\left(9.35\times10^{13}\,\frac{\text{eV}}{\text{m}}\right)\left(3.25\times10^{-15}\,\text{m}\right)=304\text{meV}\ .
$$

Alternatively, we might use the experimental value of the muonic Lamb Shift to predict the value of λ_2 in this case we find

$$
\begin{aligned}
&\Delta E_{LS}\approx\left(9.35675\times10^{13}\,\frac{\text{eV}}{\text{m}}\right)\lambda_2=206.2949\text{ meV}\\
&\Rightarrow \lambda_2=2.205\times10^{-15}\,m.
\end{aligned}
$$

This is within the range of values for λ_2 seen from nuclear phenomena.

As for the electronic Lamb Shift we find that for $\lambda_2=2\times10^{-15}\,\text{m}$ the energy difference is

$$
\Delta E_{LSe}\approx\left(5.1188\times10^{4}\,\frac{\text{eV}}{\text{m}}\right)2\times10^{-15}\,\text{m}\approx1\times10^{-7}\,\text{meV}
$$

The significance of this difference in the predicted Lamb Shift between the electronic and muonic hydrogen is that the Lamb Shift due to the non-singular potential could not be seen for electronic hydrogen; however, it becomes prominent for muonic hydrogen. This argues that

Lamb Shift measurements for muonic hydrogen, helium or other atoms would be important experiments for determining the existence of the non-singular potential and the value of λ_2.

Radiative Corrections for a Non-Singular Potential

The difference in radiative corrections between a singular and non-singular potential lays in any difference in the predicted quantum averages with respect to the atomic states. That means we need to look at

$$\left\langle \nabla^2\left(\frac{-e^2 e^{-\frac{\lambda_2}{r}}}{4\pi\varepsilon_o r}\right)\right\rangle_{at} = \frac{-e^2}{4\pi\varepsilon_o}\int d\bar{r}\,\psi_{2S}^*(\bar{r})\nabla^2\left(\frac{e^{-\frac{\lambda_2}{r}}}{r}\right)\psi_{2S}(\bar{r})$$

Any difference will come from

$$\nabla^2\left(\frac{e^{-\frac{\lambda_2}{r}}}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left(\frac{e^{-\frac{\lambda_2}{r}}}{r}\right)\right\}+0+0$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left[-\left(1-\frac{\lambda_2}{r}\right)\frac{e^{-\frac{\lambda_2}{r}}}{r^2}\right]\right\} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-\left(1-\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}\right\}$$

$$= \frac{-1}{r^2}\frac{\partial}{\partial r}\left\{\left(e^{-\frac{\lambda_2}{r}}-\frac{\lambda_2 e^{-\frac{\lambda_2}{r}}}{r}\right)\right\} = \frac{-1}{r^2}\left\{\left(\frac{\lambda_2}{r^2}e^{-\frac{\lambda_2}{r}}+\frac{\lambda_2}{r^2}\left(1-\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}\right)\right\}$$

$$= \frac{-1}{r^2}\left\{\left(\frac{\lambda_2}{r^2}\left(2-\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}\right)\right\} = \frac{\lambda_2}{r^4}\left(-2+\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}.$$

integrated over all the volume as

$$\iiint_{vol}\frac{\lambda_2}{r^4}\left(-2+\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}} = \int_0^\infty \frac{\lambda_2}{r^4}\left(-2+\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}r^2 dr\int_0^{2\pi}\sin\theta d\theta\int_{-\pi/2}^{\pi/2}d\phi$$

$$= 4\pi\int_0^\infty \frac{\lambda_2}{r^2}\left(-2+\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}}dr = 4\pi\left[-2\lambda_2\int_0^\infty\frac{e^{-\frac{\lambda_2}{r}}}{r^2}dr+\lambda_2^2\int_0^\infty\frac{e^{-\frac{\lambda_2}{r}}}{r^3}dr\right]$$

$$= 4\pi\left[-2\lambda_2\int_0^\infty\frac{e^{-\frac{\lambda_2}{r}}}{r^2}dr+\lambda_2^2\int_0^\infty\frac{e^{-\frac{\lambda_2}{r}}}{r^3}dr\right].$$

This may be evaluated by setting $u=\lambda_2/r$ so that

$$\iiint_{vol}\frac{\lambda_2}{r^4}\left(-2+\frac{\lambda_2}{r}\right)e^{-\frac{\lambda_2}{r}} = 4\pi\left[2\int_\infty^0 e^{-u}du-\int_0^\infty ue^{-u}du\right]$$

$$= 4\pi\left[2\left(-e^{-u}\Big|_\infty^0-\left(-e^{-u}(-u-1)\Big|_\infty^0\right]\right.$$

$$= 4\pi\left[2(-1+0)-(-1+0)\right]$$

$$= -4\pi$$

which is identical to the result obtained in QED for the singular potential by using the delta function.

This means there is no difference in the radiative correction between the singular and the non-singular potentials. Therefore, the expression for the radiative correction to the Lamb Shift is the same as given by the singular potential, or

$$\langle \Delta V \rangle = \frac{4}{3}\frac{e^2}{4\pi\varepsilon_o}\frac{e^2}{4\pi\varepsilon_o \hbar c}\left(\frac{\hbar}{mc}\right)^2 \frac{1}{8\pi a_o^3}\ln\left(\frac{4\varepsilon_o \hbar c}{e^2}\right)$$

where m is the orbiting mass and a_o is the Bohr orbit. This expression may be rewritten as

$$\langle \Delta V \rangle = \frac{4}{3}\frac{e^2}{4\pi\varepsilon_o}\frac{e^2}{4\pi\varepsilon_o \hbar c}\left(\frac{\hbar}{mc}\right)^2 \frac{1}{8\pi\left(\frac{4\pi\varepsilon_o \hbar^2}{me^2}\right)^3}\ln\left(\frac{4\varepsilon_o \hbar c}{e^2}\right)$$

$$= \frac{1}{6\pi}\frac{me^{10}}{\left(4\pi\varepsilon_o \hbar\right)^5 c^3}\ln\left(\frac{4\varepsilon_o \hbar c}{e^2}\right).$$

Therefore, the radiative correction varies proportional to the orbiting mass. This variation in the radiative correction term is very different from the variation of the Lamb Shift of the non-singular potential which varies as m^4. This argues that, without other factors involved, the electronic radiative correction is $4.37477 \times 10^{-3}\,\mathrm{meV}$ and the muonic radiative correction is $0.904577\,\mathrm{meV}$.

Conclusions

The first conclusion to be noted is that the non-singular potential gives an energy expression that shows separate energy levels for each quantum number *j*. However, the energy difference for the different *j* is rather small for the electronic hydrogen, but is of the order of the experimentally measured Lamb Shift for the muonic hydrogen.

On the other hand, unless there are other major contributors to the muonic radiative correction term, it would appear that the radiative correction term is dominant for the electronic hydrogen Lamb shift, but becomes a small contribution for the muonic hydrogen.

Thus the electronic Lamb shift is mostly due to the radiative correction term and the muonic Lamb Shift is mostly due to the non-singular potential itself.